Invariants under simultaneous conjugation of SL_2 matrices

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Abstract. Given tuples of SL_2-matrices, one can look at which functions in their coordinate ring do not change when we simultaneously conjugate the matrices: these are called the *invariant functions*. Our interest in this topic is motivated by the fact that these tuples occur as the so-called "monodromy group" of certain linear differential equations.

We will look at these invariant functions from three different perspectives. First, we employ classical invariant theory to find the structure of the space generated by these invariant functions. Next, we use geometric invariant theory to give a geometric interpretation of this invariant space. Finally, we place the results in the more general setting of representation theory by looking at the structure of the space of matrices as a SL_2-representation.

TODO

To be improved:

- INT: What is the representation for the fundamental group?

To be written:

- Conclusion/Discussion
- History/survey of invariant theory in general and in the case of simultaneous conjugation

Optional:

- CH1: Algebraic rather than analytic argument in the proof for $\mathbb{C}[M_0^2]^{SL_2}$.
- CH3: Add “a classification of the orbits of $Q \times Q$”?
- Semi-stable iff all invariants$\neq 0$ for $>$3 matrices: can the argument be generalized?
- Abeasis: on a minimal set of generators for the invariants of 3x3 matrices: toepassen op 2x2 matrices: gebruikt representatietheorie van eindige groepen, en deze methode wordt globaal ook gebruikt in de grotere berekeningen. Wel moeilijk te begrijpen wat er precies gebeurt. Introductie in representaties staat in Classical Invariant Theory: A Primer!

Contents

Introduction

0.1. Motivation

In this section we give a motivation for the subject of this thesis from the theory of differential equations. Understanding this motivation is not necessary to read the rest of the thesis, so the reader not intereted in differential equations may skip directly to section 0.2. The definitions in this section are mainl y from [**3**, p.113-116] and [**13**].

Suppose we are given an order 2 linear ordinary differential equation (ODE):

$$\frac{d^2y}{dt^2} + A_1(t) \cdot \frac{dy}{dt} + A_2(t) \cdot y = 0,$$

where $A_i(t)$ are fractions in t. We can consider t to be a point in the Riemann sphere $T = \mathbb{C} \cup \{\infty\}$ by substituting $u = t^{-1}$ in the case $t = \infty$. Let $S = \{t \in T \mid A_i(t) = 0 \ \exists i\}$. Then one can solve this ODE for y as an analytic function of t in a small neighborhood of any $t_0 \in T$ exactly when $t_0 \in T \backslash S$: in this case, we have a two-dimensional space of linearly independent solutions of the ODE, and we can form a basis y_1, y_2 and consider the matrix

$$M = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

Now we can consider the so-called "fundamental group" of $T \backslash S$ in the point t_0, denoted $\pi_1(T \backslash S, t_0)$. This is the group of closed loops starting at t_0 (functions $\lambda : [0,1] \to T \backslash S$ such that $\lambda(0) = \lambda(1) = t_0$) modulo homotopy. If we have n singular points, we get a free group with n generators g_i. However, if we run around all singularities once, due to the spherical structure of the Riemann sphere, this is equivalent to running around no singularities at all, so we see that

$$\pi_1(T \backslash S, t_0) = (\text{free group on } g_1, ..., g_n)/(g_1 g_2 ... g_n = 1), \tag{0.1.1}$$

or, equivalently, it is the free group on $n - 1$ generators.

Now given a point t_0 and basis M of the solution space of the ODE. Given a loop λ, we can analytically continue any solution y to the ODE along this loop, giving us a basis of the solution space at any point on the loop. Denote this basis by $M(t_0, M, \lambda, t)$. It is known that if λ is a trivial loop (i.e., one that does not run around some $s \in S$), then $M(t_0, M, \lambda, 0) = M(t_0, M, \lambda, 1)$.

However, if we run around some singularity, then generally this is not the case: $M(t_0, M, \lambda, 1)$ gives a different basis of the solution space at t_0. Since the value of $M(t_0, M, \lambda, 1)$ we get only depends on the homotopy class of the loop λ, this defines a map $\pi_1(T \backslash S, t_0) \to GL_2(\mathbb{C})$. Under suitable conditions for the ODE, one can show that in fact the $\det M = 1$, so it is a map $\pi_1(T \backslash S, t_0) \to SL_2(\mathbb{C})$.

Something not quite right here: is the map by change of basis (but then always in SL_2), or to the basis itself (but then we cannot divide out SL_2)?

Now in light of (0.1.1), we can represent $\pi_1(T \backslash S, t_0)$ as an element of

$$\{(A_1, A_2, ..., A_n) \in SL_2^n \mid A_1 A_2 ... A_n = 1\}$$

, or equivalently, as an element of $\{(A_1, A_2, ..., A_{n_1}) \in SL_2^{n-1}\}$. But as this result depends on the choice of the initial basis M, for any change of basis Q, we want to have the equivalence under simultaneous conjugation

$$(A_1, ..., A_{n-1}) \sim (QA_1Q^{-1}, ..., QA_{n-1}Q^{-1}).$$

Thus, we are motivated to look at the equivalence classes (the so-called *orbits*) under the relation $\sim$ above:

$$\{(A_1, A_2, ..., A_{n-1}) \in SL_2^{n-1}\}/\sim . \tag{0.1.2}$$

Seeing simultaneous conjugation as an action of SL_2 on SL_2^{n-1}, this is the so-called *orbit space* of SL_2^{n-1} under the action of SL_2. Interpreting this orbit space is the central topic of this thesis.

0.2. Overview

The goal of this thesis is to study the object (0.1.2). We will do this in several different ways.

First, we will use invariant theory. This means we look at the coordinate ring of the variety S of tuples of SL_2 matrices, and see what functions in this coordinate ring are invariant under SL_2, i.e., which functions give the same value for a point $p \in S$ and its conjugates $g \cdot p$ for all $g \in SL_2$. These functions thus assign a value to a given orbit, and they can be used to tell several orbits apart. The invariant functions gives rise to the algebra of invariants: an algebra consisting of all invariant functions on S. In Chapter 1, we calculate this algebra of invariants.

The space we studied using invariant theory is an affine space. However, for the remainder of the thesis, we want to work in a projective space to say more about geometric aspects of the problem. Therefore, we need to embed the space S into projective space. We can then construct a cover of large parts of this projective space and calculate the invariants on this cover. We do this in Chapter 2.

Next, we use geometric invariant theory. This will allow us to actually construct a "quotient" that assigns a geometric meaning to the notation (0.1.2) in terms of morphisms from stable and semi-stable points to a quotient space. We also compare our findings on (semi-)stability of points to the results on the algebra of invariants found earlier. This is done in Chapter 3.

Finally, we will consider the action of SL_2 on our space in the more general setting of representation theory. We describe how the action of SL_2 looks like as a representation, and see how we can apply Geometric Invariant Theory in this general setting. This is done in Chapter 4.

0.3. Basics of invariant theory

The main sources of the definitions in this section are [**11**], [**14**], [**4**, pp.30-52] and [**5**].

0.3.1. Coordinate ring. Let k be an infinite field, and W a finite-dimensional k-vector space. A function $f : W \to k$ is called *polynomial* if it is given by a polynomial in the coordinates with respect to a basis of W. Since a basis transformation causes a linear transformation of the basis vectors, this definition is independent of what basis we choose.

Now, by $k[W]$ we denote the k-algebra of polynomial functions on W; we call this the *coordinate ring* of W. If $w_1, ..., w_n$ is a basis of W and $x_1, ..., x_n$ is the dual basis of W^*, i.e., the coordinate functions, then $k[W] = k[x_1, ..., x_n]$.

This definition can be extended to the case where we have a coordinate ring not of a vector space, but of a general (affine) variety. Suppose we have an affine variety V defined as the zero set of a set of polynomials, $S \subset k[x_1, ..., x_n]$:

$$V = Z(S) = \{x \in \mathbb{A}^n \mid f(x) = 0 \ \forall f \in S\}.$$

Then define the ideal of all functions vanishing on V:

$$I(V) = \{f \in k[x_1, ..., x_n] \mid f(x) = 0 \ \forall x \in V\}.$$

The coordinate ring of the variety V, denoted $k[V]$, is then the quotient

$$k[V] = k[x_1, ..., x_n]/I(V).$$

0.3.2. The Zariski topology. One can define a topology called the *Zariski topology* on an affine space or variety. In this topology, the closed sets are defined to be the zero sets of subsets $S \subset k[x_1, ..., x_n]$:

$$V = Z(S) = \{x \in \mathbb{A}^n \mid f(x) = 0 \ \forall f \in S\}.$$

Since $Z(V) = Z(I(V))$, and every ideal in $k[x_1, ..., x_n]$ is finitely generated, without loss of assumption we can assume that the defining set S is finite. If we have a variety V, we can define its Zariski topology as the subspace topology of the affine space it is embedded in, or, equivalently, we can define its closed sets as the zero sets of subsets $S \subset k[x_1, ..., x_n]/I(V)$. The following two lemmas indicate how the Zariski topology can be used:

LEMMA 1. *Given a variety V and non-zero $f \in k[V]$. Then the set $A = \{x \in V \mid f(x) \neq 0\}$ is Zariski-dense in V.*

PROOF. Suppose the set A was contained in some closed subset $V(S) \subset V$ and let $g \in S \subset k[V]$. Then clearly $fg = 0$. But since f is non-zero, then g must be the zero function. So in fact we must have $B = V$. □

LEMMA 2. *Given a variety V and a subset $A \subset V$ Zariski-dense in V. Suppose that for some $p \in k[V]$, $p(a) = 0$ for all $a \in A$. Then $p = 0$.*

PROOF. Suppose p is not the zero function, then $A \subset V(p) \subsetneq V$, contradicting the Zariski-denseness of A. □

The Zariski topology on a projective space or variety V is simply defined by taking the subspace topology for $V \subset \mathbb{P}^n$ as a subspace of $\mathbb{A}^{n+1}$. The same description of closed sets applies as above, except that the subsets $S \subset k[x_1, ..., x_{n+1}]$ need to consist of homogeneous polynomials.

0.3.3. Representations, modules. Let W be a n-dimensional vector space over k. Suppose we have a group G and a group homomorphism $\rho : G \to GL(W) \cong GL_n(k)$, then ρ is called a *representation (of dimension n) of G on W*. This induces a linear group action of G on W via $g \cdot w = \rho(g)w$; W is then said to be a kG-module. Conversely, if we have a kG-module (that is, a linear group action of G on W), then this gives a representation by letting $\rho(g)$ be the map $w \mapsto g \cdot w$. Thus, we can use the terms "representation of G" and "kG-module" interchangeably.

Given kG-representations V, W of dimension m, n, their direct sum $V \oplus W$ and tensor product $V \otimes W$ are naturally kG-representations of dimension $m + n$, mn by the group actions

$$g \cdot (v, w) = (g \cdot v, g \cdot w); \ \ g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w).$$

A *morphism* between representations is a is a linear map $\phi : V \to W$ such that $g \cdot \phi(x) = \phi(g \cdot x)$ for all $g \in G$, $x \in V$. Two representations V, W of G are *equivalent* if this morphism is an isomorphism.

0.3.4. Invariants. Suppose W is a G-representation. A function $f \in k[W]$ is called *G-invariant* if $f(g \cdot w) = f(w)$ for all $g \in G$, $w \in W$. The invariant functions form a subalgebra of the coordinate ring of W which is called the *algebra of invariants.* We denote this by $k[W]^G$ (where the action of G on W is understood implicitly).

We can also phrase this differently: the G-action on W induces a G-action on $k[W]$ and $f \in k[W]$ is G-invariant if $g \cdot f = f$.

EXAMPLE 3. [**5**, p.3] Let $G = k^*$, $W = k^2$ with coordinate ring $k[W] = [x, y]$, $\rho(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$. Then clearly the invariant functions are linear combinations of elements of the form $x^k y^k$: $k[W]^{k^*} = k[xy]$.

Now suppose that in addition we have an affine variety defined by some polynomials $S \subset k[x_1, .., x_n]$:

$$V = Z(S) = \{x \in \mathbb{A}^n \mid f(x) = 0 \ \forall f \in S\}.$$

Suppose, also, that V is closed under G (i.e., $g \cdot v \in V$ for all $g \in G$, $v \in V$). A function $f \in k[V] = k[x_1, ..., x_n]/I(V)$ is then similarly called G-invariant if $f(gv) = f(v)$ for all $g \in G$, $v \in V$.

The equivalent formulation also holds here: the G-action on V induces a G-action on $k[V]$ by $g \cdot (f + I(V)) = g \cdot f + I(V)$ (one can easily check well-definedness), and then $f \in k[V]$ is called invariant if $g \cdot f = f$ for any $g \in G$.

The field of invariant theory, then, is concerned with finding this algebra of invariants for a given group and variety. A result in which the generators of this structure are given, is called a *first fundamental theorem* for the algebra of invariants; a result in which the defining relations are described is called a *second fundamental theorem* (if there are none, then the algebra is called *algebraically indepenent*).

0.4. Notation

In our case, we are interested in the representation we get by simultaneously conjugating tuples of matrices. In the remainder of this text, let $M_{2\times 2}$ denote the 4-dimensional vector space over $\mathbb{C}$ of 2×2 matrices. Also, let M_0 be its 3-dimensional subspace of traceless matrices, and let SL_2 be its subvariety

$$SL_2 = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_{2\times 2} \mid ad - bc = 1 \right\}.$$

Consider the action of SL_2 on $M_{2\times 2}$ by conjugation (which we will denote by $\diamond$): $g \diamond M = gMg^{-1}$, where the multiplication on the right hand side is just matrix multiplication. Note that both SL_2 and M_0 are closed under this action.

We are interested in tuples of matrices in SL_2; we write

$$SL_2^k = \{(M_1, ..., M_k) \mid M_i \in SL_2\}.$$

The action of SL_2 then extends to these tuples by simultaneous conjugation; for instance, on $SL_2^k = SL_2 \times ... \times SL_2$ the action becomes

$$g \diamond (M_1, ..., M_k) = (gM_1g^{-1}, ..., gM_kg^{-1}).$$

When we write down $k[V]^{SL_2}$ for some coordinate ring coming from a tuple of matrices, it is understood that this is the action of SL_2 on V. In this notation, the object we will study with invariant theory can be written as $\mathbb{C}[SL_2^m]^{SL_2}$.

0.5. Personal remarks

For their help in writing this thesis, I would like to thank: Jaap Top, my first supervisor, for suggesting this topic to me and helping me write the thesis; Marius van der Put, who gave the motivation for the problem and who helped with some of the calculations, and Holger Waalkens, my second supervisor.

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CHAPTER 1

Invariant theory

In [**1**], a complete description of $\mathbb{C}[M_{2\times 2}^m]^{SL_2}$ for any k by means of first and second fundamental theorems is given.

In this chapter, we adapt the methods used in [**1**] to obtain first and second fundamental theorems for $\mathbb{C}[SL_2^m]^{SL_2}$. First, in section 1.1, we indicate how we can obtain descriptions of both $\mathbb{C}[SL_2^m]^{SL_2}$ and $\mathbb{C}[M_{2\times 2}^m]^{SL_2}$ by first determining $\mathbb{C}[M_0^m]^{SL_2}$.

Next, in section 1.2, we do this for the case $m = 2$: by elementary means we can calculate $\mathbb{C}[M_0 \oplus M_0]^{SL_2}$ and then determine $\mathbb{C}[M_{2\times 2} \oplus M_{2\times 2}]^{SL_2}$ and $\mathbb{C}[SL_2 \oplus SL_2]^{SL_2}$.

For the case $m \geq 3$ we can use classical results on invariants of SO_3 matrices. In section 1.3 we give these classical results and derive descriptions of $\mathbb{C}[M_0^m]^{SL_2}$, $\mathbb{C}[M_{2\times 2}^m]^{SL_2}$ and $\mathbb{C}[SL_2^m]^{SL_2}$.

1.1. Traceless matrices

A standard trick that is often applied in the invariant theory of matrices as SL_2-representations is to write $M_{2\times 2} \cong M_0 \oplus \mathbb{C}$ as a sum of vector spaces. Now note that since SL_2 acts trivially on $\lambda\mathrm{Id}$, we can make $M_0 \oplus \mathbb{C}$ a SL_2-module by

$$g \diamond (M, t) = (gMg^{-1}, t)$$

and this will make $M^0 \oplus \mathbb{C}$ isomorphic to $M_{2\times 2}$:

Lemma 4. *Under the following isomorphism ϕ, one has $M_{2\times 2} \cong M_0 \oplus \mathbb{C}$ as SL_2-representations:*

$$M \overset{\phi}{\mapsto} (M - \frac{TrM}{2} \cdot I, TrM). \tag{1.1.1}$$

Proof. We just need to show that the following diagram commutes:

$$\begin{tikzcd} M_{2\times 2} \arrow[r, "\phi"] \arrow[d, "\alpha\diamond"] & M_0 \oplus \mathbb{C} \arrow[d, "\alpha\diamond"] \\ M_{2\times 2} \arrow[r, "\phi"] & M_0 \oplus \mathbb{C}. \end{tikzcd}$$

But this is easily seen because

$$\begin{aligned} \phi(\alpha \diamond M) &= \left(\alpha M \alpha^{-1} - \frac{\mathrm{Tr}\left(\alpha M \alpha^{-1}\right)}{2} \cdot I, \mathrm{Tr}\left(\alpha M \alpha^{-1}\right)\right) \\ &= \left(\alpha\left(M - \frac{\mathrm{Tr}M}{2} \cdot I\right)\alpha^{-1}, \mathrm{Tr}M\right) \\ &= \alpha \diamond \left(M - \frac{\mathrm{Tr}M}{2} \cdot I, \mathrm{Tr}M\right) = \alpha * \phi(M). \end{aligned}$$

□

Let us now consider the coordinate ring $\mathbb{C}[M_0 \oplus \mathbb{C}] = \mathbb{C}[M_0] \otimes \mathbb{C}[t]$. Note that SL_2 acts on $\mathbb{C}[M_0]$ in the usual way, and it acts trivially on $\mathbb{C}[t]$. But then clearly we must have

$$\mathbb{C}[M_0 \oplus \mathbb{C}]^{SL_2} = \mathbb{C}[M_0]^{SL_2} \otimes \mathbb{C}[t].$$

We note that the isomorphism ϕ also induces an isomorphism between the coordinate rings $\mathbb{C}[M_0 \oplus \mathbb{C}]$ and $\mathbb{C}[M_{2\times 2}]$, so:

$$\mathbb{C}[M_{2\times 2}]^{SL_2} \cong \mathbb{C}[M_0]^{SL_2} \otimes \mathbb{C}[t]. \tag{1.1.2}$$

We can also generalize the reasoning by which we obtained (1.1.2) to tuples of matrices. Let us first introduce some notation for the coordinate rings we will use throughout this chapter.

For M_0^m, let a_i, b_i, c_i for $1 \leq i \leq m$ be the coordinate functions of this vector space forming generic matrices

$$y_i = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix}.$$

Thus, we can talk about $\mathbb{C}[M_0^m]$, $\mathbb{C}[a_1, b_1, c_1, ..., a_m, b_m, c_m]$ or $\mathbb{C}[y_1, ..., y_m]$ interchangeably. We denote by t_i for $1 \leq i \leq m$ the coordinate vectors of the $\mathbb{C}$'s we get from (1.1.2).

For $M_{2\times 2}^m$, let a_i', b_i', c_i', d_i' for $1 \leq i \leq m$ be the coordinate functions forming generic matrices

$$x_i = \begin{pmatrix} a_i' & b_i' \\ c_i' & d_i' \end{pmatrix}.$$

For SL_2^m, the notation is the same as in the case of $M_{2\times 2}^m$, except that we are working in a subvariety. So the coordinate ring we consider is

$$\mathbb{C}[SL_2^m] = \mathbb{C}[x_1, ..., x_m]/(I); \quad (I) = (\det x_1 - 1, ..., \det x_m - 1).$$

So, what happens in the case $m > 1$? Lemma 4 is obviously also true for tuples, so we have the following isomorphism:

$$\begin{array}{rcl} M_{2\times 2}^m & \to & M_0^m \oplus \mathbb{C}^{\oplus m} \\ (M_1, ..., M_m) & \mapsto & (M_1 - \frac{1}{2}\mathrm{Tr}M_1 \cdot I, ..., \mathrm{Tr}M_1, ..., \mathrm{Tr}M_m). \end{array} \tag{1.1.3}$$

This induces $\mathbb{C}[M_{2\times 2}^m] \cong \mathbb{C}[M_0^m] \otimes \mathbb{C}[t_1, ..., t_m]$, and, reasoning as above, we immediately obtain the following result:

Proposition 5. *We have an isomorphism of SL_2-modules:*

$$\mathbb{C}[M_{2\times 2}^m]^{SL_2} \cong \mathbb{C}[M_0^m]^{SL_2} \otimes \mathbb{C}[t_1, ..., t_m].$$

The identification is induced by (1.1.3).

In the SL_2 case, the result is slightly more involved:

Proposition 6. *We have an isomorphism of SL_2-modules:*

$$\mathbb{C}[SL_2^m]^{SL_2} \cong \left(\mathbb{C}[M_0^m]^{SL_2} \otimes \mathbb{C}[t_1, ..., t_m]\right)/(I'),$$

where

$$(I') = (dety_1 + \frac{1}{4}t_1^2 - 1, ..., dety_m + \frac{1}{4}t_m^2 - 1).$$

The identification is induced by (1.1.3).

Proof. We have the same generators of the coordinate ring as with $M_{2\times 2}$, but they are divided out by the ideal

$$(I) = (\det x_1 - 1, ..., \det x_m - 1).$$

One verifies that under ϕ, this ideal gets translated to the ideal

$$(I') = (\det y_1 + \frac{1}{4}t_1^2 - 1, ..., \det y_m + \frac{1}{4}t_m^2 - 1),$$

so we have the SL_2-module isomorphism

$$\mathbb{C}[SL_2^m] \cong \left(\mathbb{C}[M_0^m] \otimes \mathbb{C}[t_1, ..., t_m]\right)/I'.$$

Now by eliminating higher powers of the t_i using the ideal, we can write

$$\mathbb{C}[SL_2^m] \cong \mathbb{C}[M_0^m] \otimes (1 \oplus t_1) \otimes ... \otimes (1 \oplus t_m),$$

and again, we note that SL_2 just acts on $\mathbb{C}[M_0^m]$, so

$$\mathbb{C}[SL_2^m]^{SL_2} \cong \mathbb{C}[M_0^m]^{SL_2} \otimes (1 \oplus t_1) \otimes ... \otimes (1 \oplus t_m).$$

Introducing the ideal (I') again we get our claim. □

We note some identities that will be useful later on:

Lemma 7. *Under the isomorphism* $\phi : \mathbb{C}[M_0^m] \otimes \mathbb{C}[t_m, ..., t_m] \to \mathbb{C}[M_{2\times 2}^m]$ *induced by (1.1.3), we have*

$$\begin{array}{rcl} \phi(t_i) & = & Trx_i \\ \phi(Try_iy_j) & = & Trx_ix_j - \frac{1}{2}Trx_i\,Trx_j \\ \phi(Tr(y_iy_jy_k - y_iy_ky_j)) & = & Tr(x_ix_jx_k - x_ix_kx_j). \end{array}$$

Proof. One verifies this just by direct computation, e.g.:

$$\begin{array}{rcl} \phi(\mathrm{Tr}y_iy_j) & = & \mathrm{Tr}((x_i - \frac{1}{2}\mathrm{Tr}x_i \cdot I) \cdot (x_j - \frac{1}{2}\mathrm{Tr}x_j \cdot I) \\ & = & \mathrm{Tr}(x_ix_j - \frac{1}{2}\mathrm{Tr}\,(x_j) \cdot x_i - \frac{1}{2}\mathrm{Tr}\,(x_i)\,x_j + \frac{1}{4}\mathrm{Tr}x_i\mathrm{Tr}x_j \cdot I) \\ & = & \mathrm{Tr}x_ix_j - \frac{1}{2}\mathrm{Tr}x_i\mathrm{Tr}x_j. \end{array}$$

□

1.2. Two 2×2 matrices

Now, we will calculate the algebra of invariants for $M_0 \oplus M_0$. This proof is a more detailed version of the one found in [**5**, p.21]. We will prove the following:

Proposition 8. $\mathbb{C}[M_0 \oplus M_0]^{SL_2} = \mathbb{C}[Try_1y_2, Try_1^2, Try_2^2]$, *and these generators are algebraically independent.*

Proof. First of all, we prove algebraic independence. Consider the following evaluation homomorphism:

$$\begin{array}{rcl} \mathbb{C}[a_1, b_1, c_1, a_2b_2, c_2] & \xrightarrow{\psi} & \mathbb{C}[\nu, \mu, \lambda] \\ a_1 & \mapsto & \nu \\ b_1 & \mapsto & \mu \\ a_2 & \mapsto & \lambda \\ c_1, b_2, c_2 & \mapsto & 0. \end{array} \tag{1.2.1}$$

Note that under this homomorphism, Tr $y_1^2 \mapsto 2\nu^2$; $\mathrm{Tr}y_2^2 \mapsto 2\lambda$; $\mathrm{Tr}y_1y_2 \mapsto \mu$. Suppose there was was a non-trivial relation between $\mathrm{Tr}A^2$, $\mathrm{Tr}B^2$, $\mathrm{Tr}AB$: then this would translate under ψ to a non-trivial relation between $2\nu^2$, 2λ, μ. But these three elements are clearly algebraically independent in $\mathbb{C}[\nu, \mu, \lambda]$, giving a contradiction.

Next we prove that any invariant function is a polynomial in these three traces. So let $F \in \mathbb{C}[a_1, b_1, c_1, a_2, b_2, c_2]$ be an invariant function. Let us consider the following evaluation homomorphism:

$$\begin{array}{rcl} \mathbb{C}[a_1, b_1, c_1, a_2, b_2, c_2] & \xrightarrow{\phi} & \mathbb{C}[a, c, t] \\ a_1 & \mapsto & t \\ b_1, c_1 & \mapsto & 0 \\ a_2 & \mapsto & a \\ b_2 & \mapsto & 1 \\ c_2 & \mapsto & c. \end{array}$$

Hence we get $\phi(F) \in \mathbb{C}[a, c, t]$. But now notice that for any $a, c \neq 0, t \neq 0 \in \mathbb{C}$, we have

$$\begin{pmatrix} 0 & 1/\sqrt{c} \\ \sqrt{c} & 0 \end{pmatrix} \diamond \left(\begin{pmatrix} t & \\ & -t \end{pmatrix}, \begin{pmatrix} a & 1 \\ c & -a \end{pmatrix} \right) = \left(\begin{pmatrix} -t & \\ & t \end{pmatrix}, \begin{pmatrix} -a & 1 \\ c & a \end{pmatrix} \right).$$

Because F is invariant under conjugation, evaluation of $\phi(F)$ in the second matrix pair must give the same result as evaluating in the first matrix pair, ie, we have

$$\phi(F)(a, c, t) = \phi(F)(-a, c, -t)$$

in the ring $\mathbb{C}[a, c, t]$. Hence $\phi(F)$ is a polynomial in a^2, at, t^2 and c. But we note the following identities:

$$t^2 = \phi(\frac{1}{2}\mathrm{Tr} y_1^2); \quad a^2 + c = \phi(\frac{1}{2}\mathrm{Tr} y_2^2); \quad at = \phi(\frac{1}{2}\mathrm{Tr} y_1 y_2);$$

$$\phi(\mathrm{Tr} y_1^2)c = \phi(\frac{1}{2}\mathrm{Tr} y_1^2 \mathrm{Tr} y_2^2 - (\mathrm{Tr} y_1 y_2)^2),$$

so that for certain $k \geq 0$ and $P \in \mathbb{C}[\mathrm{Tr} y_1^2, \mathrm{Tr} y_2^2, \mathrm{Tr} y_1 y_2]$ we have

$$\phi((\mathrm{Tr} y_1^2)^k) \cdot \phi(F) = \phi(P). \tag{1.2.2}$$

We need the fact that the set

$$\begin{aligned} M' \quad = \quad & \{(A, B) \in M_0^2 \mid \exists Q \in SL_2, a, c \neq 0, t \neq 0 \in \mathbb{C} : \\ & Q(A, B)Q^{-1} = \left(\left(\begin{smallmatrix} t & \\ & -t \end{smallmatrix}\right), \left(\begin{smallmatrix} a & 1 \\ c & -a \end{smallmatrix}\right) \right) \} \end{aligned}$$

is Zariski-dense in M_0^2. We prove this rather technical fact in Lemma 9 below. Now take any $p = (A, B) \in M'$, then p is conjugate to a point $p' = \left(\left(\begin{smallmatrix} t & 0 \\ 0 & -t \end{smallmatrix}\right), \left(\begin{smallmatrix} a & 1 \\ c & -a \end{smallmatrix}\right) \right) \in M'$ in diagonal form, so:

$$\begin{array}{rcll} ((\mathrm{Tr} A^2)^k F)(p) & = & ((\mathrm{Tr} A^2)^k F)(p') & \text{(invariance of } F\text{)} \\ & = & \phi((\mathrm{Tr} A^2)^k)\phi(F)(a, c, t) & \text{(definition of } \phi\text{)} \\ & = & \phi(P)(a, c, t) & \text{(1.2.2)} \\ & = & P(p') & \text{(definition of } \phi\text{)} \\ & = & P(p) & \text{(invariance of } P\text{),} \end{array}$$

but then by Lemma 2, the Zariski-denseness of M' implies that this holds as a polynomial equation on the whole of M_0^2:

$$(\mathrm{Tr} y_1^2)^k F = P. \tag{1.2.3}$$

Thus, so far, we have established that

$$F \in \mathbb{C}[\mathrm{Tr} y_1^2, \mathrm{Tr} y_2^2, \mathrm{Tr} y_1 y_2, \frac{1}{\mathrm{Tr} y_1^2}] \cap \mathbb{C}[a_1, b_1, c_1, a_2, b_2, c_2].$$

Now on the one hand, F is just a polynomial 6 variables; on the other hand, it can be a fraction in $\mathrm{Tr} y_1^{-2}$ which would have some asymptotic behaviour. We would like to conclude that in fact this fraction cannot occur, so we have a polynomial $F \in \mathbb{C}[\mathrm{Tr} y_1^2, \mathrm{Tr} y_2^2, \mathrm{Tr} y_1 y_2]$, finishing our proof.

First, write F as a Laurent polynomial in $\mathrm{Tr} y_1^2$:

$$F = \sum_{i=i_0}^{k} f_i \cdot \left(\mathrm{Tr} y_1^2\right)^k ; \;\; f_i \in \mathbb{C}[\mathrm{Tr} y_1 y_2, \mathrm{Tr} y_2^2],$$

and suppose that i_0 is negative. If we apply the evaluation homomorphism ψ from (1.2.1) to F, we get

$$\psi(F) = \sum_{i=i_0}^{k} f_i' \cdot \left(2\nu^2\right)^k \in \mathbb{C}[\mu, \lambda](\nu); \;\; f_i' = \psi(f_i) \in \mathbb{C}[\mu, \lambda].$$

Now, since f_i was nonzero, we can choose $\hat{\mu}$, $\hat{\lambda}$ so that $f_{i_0}(\hat{\mu}, \hat{\lambda}) \neq 0$. Now apply the evaluation homomorphism $\chi : \;\mu \mapsto \hat{\mu}; \; \lambda \mapsto \hat{\lambda}$, then

$$\chi(\psi(F)) = \sum_{i=i_0}^{k} f_i'(\hat{\mu}, \hat{\lambda}) \cdot \left(2\nu^2\right)^k \in \mathbb{C}(\nu),$$

so we have constructed a function in $\chi(\psi(F)) \in \mathbb{C}(\nu)$ which by construction tends to $\pm\infty$ as ν tends to 0.

However, we also had $F \in \mathbb{C}[a_1, b_1, c_1, a_2, b_2, c_2]$, so in fact $\psi(F) \in \mathbb{C}[\nu, \mu, \lambda]$ and $\chi(\psi(F)) \in \mathbb{C}[\nu]$! This contradicts the asymptotic behaviour that we just established! So in fact, i_0 cannot be negative, and $F \in \mathbb{C}[\mathrm{Tr} y_1^2, \mathrm{Tr} y_2^2, \mathrm{Tr} y_1 y_2]$ as we wanted to show. ☐

We are just left with the technical lemma we had:

LEMMA 9. *The set*

$$\begin{aligned} S \;\; = \;\; & \{(A,B) \in M_0^2 \mid \exists Q \in SL_2, a, c \neq 0, t \neq 0 \in \mathbb{C} : \\ & Q(A,B)Q^{-1} = \left(\left(\begin{smallmatrix} t & \\ & -t \end{smallmatrix}\right), \left(\begin{smallmatrix} a & 1 \\ c & -a \end{smallmatrix}\right)\right)\} \end{aligned}$$

is Zariski-dense in M_0^2.

PROOF. First let

$$S' = \{(A,B) \in M_0^2 \mid A \text{ has two distinct eigenvalues}\}.$$

It is standard that this set is Zariski-dense in S: as a defining equation for Lemma 1 we can use the fact that the discriminant of its characteristic polynomial must be non-zero.

Now we consider the subset

$$S'' = \{(A,B) \in S' \mid \mathrm{Tr} A^2 \mathrm{Tr} B^2 - (\mathrm{Tr} AB)^2 \neq 0\}.$$

This set is obviously Zariski-dense in S' and hence in S; we claim that this is the set we need.

For let us consider a tuple (A, B) where A has distinct eigenvalues. Then for some matrix Q,

$$Q(A,B)Q^{-1} = \left(\begin{pmatrix} t & \\ & -t \end{pmatrix}, \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right); \;\; t \neq 0.$$

Now,

$$\begin{aligned} \mathrm{Tr} A^2 \mathrm{Tr} B^2 - (\mathrm{Tr} AB)^2 \;\; &= \;\; \left(2t^2\right)\left(2a^2 + 2bc\right) - (2at)^2 \\ &= \;\; 4a^2t^2 + 4t^2bc - 4a^2t^2 = 4t^2bc. \end{aligned}$$

Then for this to be non-zero, we need $b, c \neq 0$.

But now we note that in choosing Q to diagonalize A, we have a degree of freedom that enables us to multiply b with a non-zero constant μ^2; c then gets multiplied with μ^{-2}. So in fact we can make $b = 1$, $c \neq 0$, so $(A, B) \in S$, and we are done. ☐

But now Propositions 5 and 6 allow us to translate these results to $\mathbb{C}[M_{2\times 2}^2]^{SL_2}$ and $\mathbb{C}[SL_2^2]^{SL_2}$:

Corollary 10. *$\mathbb{C}[M_{2\times 2}\oplus M_{2\times 2}]^{SL_2} = \mathbb{C}[Trx_1, Trx_2, Trx_1x_2, Trx_1^2, Trx_2^2]$, and these generators are algebraically independent.*

Proof. By Proposition 5,

$$\begin{aligned} \mathbb{C}[M_{2\times 2}\oplus M_{2\times 2}]^{SL_2} &\cong \mathbb{C}[M_0^m]^{SL_2}\otimes \mathbb{C}[t_1]\otimes \mathbb{C}[t_2]. \\ &= \mathbb{C}[\mathrm{Tr}y_1y_2, \mathrm{Tr}y_1^2, \mathrm{Tr}y_2^2]\otimes \mathbb{C}[t_1].\otimes \mathbb{C}[t_2]. \end{aligned}$$

Now, translating this ring under the isomorphism (1.1.3) and using Lemma 4, we get the required result. □

Corollary 11. *$\mathbb{C}[SL_2\oplus SL_2]^{SL_2} = \mathbb{C}[Trx_1, Trx_2, Trx_1x_2]$, and these generators are algebraically independent.*

Proof. By Proposition 6, letting $(I') = (\mathrm{det}y_1 + \frac{1}{4}t_1^2 - 1, \mathrm{det}y_2 + \frac{1}{4}t_2^2 - 1)$, we have:

$$\begin{aligned} \mathbb{C}[SL_2^2]^{SL_2} &\cong \left(\mathbb{C}[M_0^2]^{SL_2}\otimes \mathbb{C}[t_1]\otimes \mathbb{C}[t_2]\right)/(I'), \\ &= \left(\mathbb{C}[\mathrm{Tr}y_1y_2, \mathrm{Tr}y_1^2, \mathrm{Tr}y_2^2]\otimes \mathbb{C}[t_1]\otimes \mathbb{C}[t_2]\right)/(I'), \\ &\cong \mathbb{C}[\mathrm{Tr}y_1y_2, t_1, t_2], \end{aligned}$$

where for the last isomorphism we eliminated $\mathrm{Tr}y_i^2$ by noting the following equality in $\left(\mathbb{C}[\mathrm{Tr}y_1y_2, \mathrm{Tr}y_1^2, \mathrm{Tr}y_2^2]\otimes \mathbb{C}[t_1]\otimes \mathbb{C}[t_2]\right)/(I')$:

$$\mathrm{Tr}y_i^2 = -2\det y_i = \frac{1}{2}t_i^2 - 2.$$

Using the isomorphism (1.1.3) and Lemma 4, we get the required result. □

1.3. The general case

We now calculate $\mathbb{C}[M_0^k]^{SL_2}$ for any $k \geq 2$. We do this by using classical results on invariants of SO_3-representations.

First we introduce these classical results; in the second subsection, we use them to determine $\mathbb{C}[M_0^k]^{SL_2}$. The approach taken here is a slight adaptation from the the one presented in [**1**]. In the third subsection, we apply these results, much like before, to find $\mathbb{C}[M_{2\times 2}^k]^{SL_2}$ and $\mathbb{C}[SL_2^k]^{SL_2}$

1.3.1. Invariants of orthorgonal groups.

Let V_n be a n-dimensional vector space with basis $\{e_1, ..., e_n\}$. Define the non-degenerate, symmetric bilinear form called the *dot product* by:

$$\left\langle \sum v_ie_i, \sum w_ie_i\right\rangle = \sum v_iw_i.$$

Let O_n be the group of matrices $g\in GL(V_n)$ so that $\langle g\cdot x, g\cdot x\rangle = \langle x, x\rangle$ for all $x\in V_n$; this is well known to be equivalent to the matrix satisfying $G^TG = I$. These matrices have determinant ± 1; the subgroup SO_n consists of those matrices with determinant 1.

Now for vectors $v_i = (v_{i1}, .., v_{in})$ and $w_i = (w_{i1}, .., w_{in})$, we define

$$\Delta_n(v_1, v_2, .., v_n) = \det(v_1, .., v_n) = \begin{vmatrix} v_{11} & v_{21} & \cdots & v_{n1} \\ v_{12} & & & v_{n2} \\ \vdots & & & \vdots \\ v_{1n} & v_{2n} & \cdots & v_{nn} \end{vmatrix};$$

$$\Gamma_k(v_1,...,v_k|w_1,...,w_k) = \det\left(\langle v_i, w_j\rangle\,|_{i,j=1,\dots,k}\right) = \begin{vmatrix} \langle v_1, w_1\rangle & \cdots & \langle v_1, w_k\rangle \\ \langle v_2, w_1\rangle & & \langle v_2, w_k\rangle \\ \vdots & & \vdots \\ \langle v_k, w_1\rangle & \cdots & \langle v_k, w_k\rangle \end{vmatrix}.$$

The vector space V_n^m is a SO_n- or O_n-representation by letting the groups act diagonally on m copies of V_n by matrix multiplication, and the first and second fundamental theorems of invariant theory for $\mathbb{C}[V_n^m]^{O_n}$ and $\mathbb{C}[V_n^m]^{SO_n}$ are well-known; for example, in his classical work [**10**], Weyl gives both the First Fundamental Theorem ([**10**, p.53]), and the Second Fundamental Theorem ([**10**, p.75]), albeit he works only over the real numbers. In [**1**], we find them stated as follows.

We introduce the following notation: let $v_{i,j}$ be the coordinate functions for the ith copy of $V_n$in V_n^m.

THEOREM 12. *(The fundamental theorems for $\mathbb{C}[V_n^m]^{O_n}$) Consider $\mathbb{C}[V_n^m]^{O_n}$.*

(i) Its algebra of invariants is generated by $\langle v_i, v_j\rangle$ for $1 \le i, j \le m$;

(ii) the defining relations for the algebra of invariants are ($1 \le i_0 < i_1 < ... < i_n \le m$; $1 \le j_0 < j_1 < ... < j_n \le m$):

$$\Gamma_{n+1}(v_{i_0}, v_{i_1}, ..., v_{i_n}|v_{j_0}, v_{j_1}, ..., v_{j_n}) = 0.$$

Similarly for SO_n:

THEOREM 13. *(The fundamental theorems for $\mathbb{C}[V_n^m]^{SO_n}$) Consider $\mathbb{C}[V_n^m]^{SO_n}$.*

(i) Its algebra of invariants is generated by the generators of $\mathbb{C}[V_n^m]^{O_n}$ and $\Delta(v_{i_1}, v_{i_2}, ..., v_{i_n})\mathbb{C}[V_n^m]^{O_n}$, $1 \le i_1 < ... < i_n \le m$;

(ii) the defining relations for the algebra of invariants are

$$\Delta_n(u_{i_1}, ..., u_{i_n})\Delta_n(u_{j_1}, ..., u_{j_n}) - \Gamma_n(u_{i_1}, ..., u_{i_n}|u_{j_1}, ..., u_{j_n}) = 0$$

$$(1 \le i_1 < ... < i_n < m;\ 1 \le j_1 < ... < j_n \le m);$$

$$\sum_{k=0}^{n}(-1)^k \langle u_i, u_{j_k}\rangle \Delta_n(u_{j_0}, ..., \widehat{u_{j_k}}, ..., u_{j_n}) = 0$$

$$1 \le j_0 < j_1 < ... < j_n \le m$$

(here, $\widehat{u_{j_k}}$ means that this u_{j_k} is removed from the expression).

Note that for SO_n we do not need the defining relations for O_n: it turns out that these relations already follow from the relations mentioned in the Theorem for SO_n.

1.3.2. Connecting the actions of SO_3 and SL_2.

In section 1.3.1, we considered the invariant functions on V_n^m under the action of SO_n.

Let us now look at the the case of SO_3 acting on V_3^m. We can identify this vector space with the space M_0^m, and fix basis

$$\left\{ e_{i,1} = \frac{1}{2}\sqrt{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right),\ e_{i,2} = \frac{1}{2}\sqrt{2}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right),\ e_{i,3} = \frac{1}{2}\sqrt{-2}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \right\},\ 1 \le i \le m;$$

one can now check that for the bilinear form we have$\langle u, v\rangle = \mathrm{Tr}uv$.

Our representation of SL_2 on M_0^m by simultaneous conjugation is now simply a group homomorphism $\phi : SL_2 \to GL(W)$. We claim that this map is a surjection to SO_3. But then $\mathbb{C}[M_0^m]^{SL_2} = \mathbb{C}[V_3^m]^{SO_3}$, and we just have to translate the invariant theory of SO_3 to the SL_2 case. First our claim:

LEMMA 14. *The mapping ϕ above is a surjective map $SL_2 \to SO_3$.*

PROOF. Note that we know that the action of SL_2 (which we denote by $\diamond$) preserves the dot product, because the dot product of u and v is the same as the trace of uv, which is well-known to be invariant under conjugation.

We will show the following intermediate result:

$$\forall M \in SO_3 \ \exists Q \in SL_2 : \ Q \diamond e_1 = Me_1, \, Q \diamond e_2 = Me_2. \tag{1.3.1}$$

For suppose we have shown (1.3.1). Denote $f_i = Me_i$, and note that because the dot product is preserved under the action of M, we have $\langle f_i, f_j \rangle = \delta_{ij}$. Then:

$$\begin{aligned} Q \diamond e_3 &= \langle Q \diamond e_3, f_1 \rangle f_1 + \langle Q \diamond e_3, f_2 \rangle f_2 + \langle Q \diamond e_3, f_3 \rangle f_3 \\ &= \langle Q \diamond e_3, Q \diamond e_1 \rangle f_1 + \langle Q \diamond e_3, Q \diamond e_2 \rangle f_2 + \langle Q \diamond e_3, f_3 \rangle f_3 \\ &= \langle e_3, e_1 \rangle f_1 + \langle e_3, e_2 \rangle f_2 + \langle Q \diamond e_3, f_3 \rangle f_3 = \langle Q \diamond e_3, f_3 \rangle f_3. \end{aligned}$$

But also:

$$1 = \langle e_3, e_3 \rangle = \langle Q \diamond e_3, Q \diamond e_3 \rangle = \langle Q \diamond e_3, f_3 \rangle^2 .$$

So then either $\phi(Q) = M \in SO_3$, or $\phi(Q) = M' \in O_3 \backslash SO_3$, where M' is M but with the last row multiplied by -1. We then show that in fact $\det \phi(Q) = 1$ holds for any $Q \in SL_2$, which will conclude our proof.

Now we prove the statement (1.3.1). Since by assumption, $\langle f_1, f_1 \rangle = 1$, we have that f_1 must be diagonalizable. For otherwise conjugating it into Jordan form gives one of the non-diagonalizable matrices, which have norm 0:

$$\left\langle \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \right\rangle = 0, \text{ or } \langle 0, 0 \rangle = 0.$$

So the Jordan form of f_1 has to be

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \text{ for some } \lambda \neq 0,$$

and then preservation of the norm gives us that in fact f_1 must diagonalize to e_1, ie, $Q^{-1} f_1 Q = e_1$ for some $Q \in SL_2$; thus, $Q^{-1} \diamond e_1 = f_1$.

Now, look at the action of Q on f_2: let's suppose $Q \diamond f_2 = ae_2 + be_3$ (we have $\langle Q \diamond f_2, e_1 \rangle = 0$ by the above). Note that we can always find α such that

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \diamond e_2 = ae_2 + be_3 :$$

a calculation shows that this conjugation is equal to $\frac{1}{2}((\alpha^2 + \alpha^{-2})e_2 - i(\alpha^2 - \alpha^{-2})e_3)$, so one sees that choosing $\alpha = \sqrt{a - bi}$ does the trick.

But then note that

$$\left(Q^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \right) \diamond e_2 = Q^{-1} \diamond \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \diamond e_2 = Q^{-1} \diamond (ae_2 + be_3) = f_2,$$

and, since conjugating with this diagonal matrix does not change e_1, we see that the matrix

$$Q' = Q^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in SL_2$$

has the property $Q'e_1 = f_1$, $Q'e_2 = f_2$, as requested. This proves (1.3.1).

We now show that $\det \phi(M) = 1$ for any diagonalizable $M \in SL_2$. Let us say that M can be diagonalized with matrix S to $\operatorname{diag}(\lambda, \lambda^{-1})$. Then:

$$\begin{aligned} \det \phi(M) &= \det \phi \left(S \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} S^{-1} \right) \\ &= \det \phi(S) \det \phi \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \det \phi(S)^{-1} = \det \phi \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} . \end{aligned}$$

We already saw how such a diagonal matrix acts on e_2; a calculation similarly shows that it sends e_3 to $\frac{1}{2}(-i(\lambda^2 - \lambda^{-2})e_2 + (\lambda^2 + \lambda^{-2})e_3)$, so we get the following

$$\det \phi \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(\lambda^2 + \lambda^{-2}) & \frac{1}{2}i(\lambda^2 - \lambda^{-2}) \\ 0 & -\frac{1}{2}i(\lambda^2 - \lambda^{-2}) & \frac{1}{2}(\lambda^2 + \lambda^{-2}) \end{vmatrix} = 1.$$

We note that $\det \phi(M)$ is a homogeneous polynomial of degree 6 in the coordinate functions of M: the elements of the O_3-matrix are products of the entries of M and M^{-1} and M^{-1} does not have fractions because M has determinant 1. So, we have that $\det \phi(M) - 1 = 0$ for all M in the set of diagonalizable matrices in SL_2.

But this set is Zariski-dense in SL_2 because its subset consisting of matrices with two different eigenvalues is already dense in SL_2 (this can be expressed in a polynomial equation by saying that the discriminant of the characteristic polynomial is nonzero, then use Lemma 1). So, in fact $\det \phi(M) = 1$ for all $m \in SL_2$ by Lemma 2, completing the proof. ☐

1.3.3. The structure of $\mathbb{C}[M_0^m]^{SL_2}$, $\mathbb{C}[M_{2\times 2}^m]^{SL_2}$, $\mathbb{C}[SL_2^m]^{SL_2}$. Now, as argued, Theorem 13 directly gives us $\mathbb{C}[M_0^m]^{SL_2}$; we just re-formulate it to get rid of the dot product notation. As generators,we have $\langle y_i, y_j \rangle = \mathrm{Tr} y_i y_j$ and $\Delta(y_{i_1}, y_{i_2}, y_{i_3})$. We can also write this Δ in terms of traces, for one verifies that the following relation holds:

$$\Delta(y_{i_1}, y_{i_2}, y_{i_3}) = -\frac{1}{\sqrt{-2}}\mathrm{Tr}(y_{i_1} y_{i_2} y_{i_3} - y_{i_1} y_{i_3} y_{i_2}). \tag{1.3.2}$$

For the first type of relation given in the theorem, we note that

$$\begin{aligned} \Gamma_k(y_{i_1}, y_{i_2}, y_{i_3} | y_{j_1}, y_{j_2}, y_{j_3}) &= \begin{vmatrix} \langle y_{i_1}, y_{j_1} \rangle & \langle y_{i_1}, y_{j_2} \rangle & \langle y_{i_1}, y_{j_3} \rangle \\ \langle y_{i_2}, y_{j_1} \rangle & \langle y_{i_2}, y_{j_2} \rangle & \langle y_{i_2}, y_{j_3} \rangle \\ \langle y_{i_3}, y_{j_1} \rangle & \langle y_{i_3}, y_{j_2} \rangle & \langle y_{i_3}, y_{j_3} \rangle \end{vmatrix} \\ &= \begin{vmatrix} \mathrm{Tr} y_{i_1} y_{j_1} & \mathrm{Tr} y_{i_1} y_{j_2} & \mathrm{Tr} y_{i_1} y_{j_3} \\ \mathrm{Tr} y_{i_2}, y_{j_1} & \mathrm{Tr} y_{i_2} y_{j_2} & \mathrm{Tr} y_{i_2} y_{j_3} \\ \mathrm{Tr} y_{i_3}, y_{j_1} & \mathrm{Tr} y_{i_3} y_{j_2} & \mathrm{Tr} y_{i_3} y_{j_3} \end{vmatrix}. \end{aligned}$$

The second type of relation directly translates using (1.3.2). Hence, we get:

THEOREM 15. *Consider* $\mathbb{C}[M^0 \oplus ... \oplus M^0]^{SL_2}$, *where* SL_2 *acts by simultaneous conjugation on* m *matrices. Then*

(i) this algebra of invariants is generated by $Try_j y_k$,$1 \leq j \leq k \leq m$, *and*

$$s_3(y_{j_1}, y_{j_2}, y_{j_3}) := Tr(y_{j_1}, y_{j_2}, y_{j_3} - y_{j_1}, y_{j_3}, y_{j_2});\ 1 \leq j_1 < j_2 < j_3 \leq m;$$

(ii) the defining relations for the invariant ring are

$$Trs_3(y_{i_1} y_{i_2} y_{i_3}) \cdot Trs_3(y_{j_1} y_{j_2} y_{j_3}) + 2 \begin{vmatrix} Try_{i_1} y_{j_1} & Try_{i_1} y_{j_2} & Try_{i_1} y_{j_3} \\ Try_{i_2}, y_{j_1} & Try_{i_2} y_{j_2} & Try_{i_2} y_{j_3} \\ Try_{i_3}, y_{j_1} & Try_{i_3} y_{j_2} & Try_{i_3} y_{j_3} \end{vmatrix} = 0;$$

$1 \leq i_1 < i_2 < i_3 \leq m$; $1 \leq j_1 < j_2 < j_3 \leq m$; *and*

$$\sum_{k=0}^{3} (-1)^k\, Try_i y_{p_k}\, Trs_3(y_{p_0}, ..., \widehat{y_{p_k}}, ..., y_{p_3}) = 0;$$

$1 \leq p_0 < p_1 < p_2 < p_3 \leq m$. *Here,* $\hat{x}$ *means* x *is removed from the expression.*

As before, a rather straightforward application of Proposition 5 and Lemma 7 now directly gives us:

COROLLARY 16. *Consider* $\mathbb{C}[M_{2\times 2} \oplus ... \oplus M_{2\times 2}]^{SL_2}$*, where* SL_2 *acts by simultaneous conjugation on* m *matrices. Then*

(i) this algebra of invariants is generated by $Tr x_j$*,* $1 \leq j \leq m$*,* $Tr x_j x_k$*,*$1 \leq j \leq k \leq m$*, and*

$$s_3(x_{j_1}, x_{j_2}, x_{j_3}) := Tr(x_{j_1}, x_{j_2}, x_{j_3} - x_{j_1}, x_{j_3}, x_{j_2});\ 1 \leq j_1 < j_2 < j_3 \leq m;$$

(ii) the defining relations for the invariant ring are

$$Trs_3(x_{i_1}x_{i_2}x_{i_3}) \cdot Trs_3(x_{j_1}x_{j_2}x_{j_3}) + \\ 2\begin{vmatrix} Trx_{i_1}x_{j_1} - \frac{1}{2}Trx_{i_1}Trx_{j_1} & \cdots & Trx_{i_1}x_{j_3} - \frac{1}{2}Trx_{i_1}Trx_{j_3} \\ \vdots & \ddots & \vdots \\ Trx_{i_3}x_{j_1} - \frac{1}{2}Trx_{i_3}Trx_{j_1} & \cdots & Trx_{i_3}x_{j_3} - \frac{1}{2}Trx_{i_3}Trx_{j_3} \end{vmatrix} = 0;$$

$1 \leq i_1 < i_2 < i_3 \leq m$*;* $1 \leq j_1 < j_2 < j_3 \leq m$*; and*

$$\sum_{k=0}^{3}(-1)^k\left(Trx_ix_{p_k} - \frac{1}{2}Trx_i\,Trx_{p_k}\right)Trs_3(y_{p_0}, ..., \widehat{y_{p_k}}, ..., y_{p_3}) = 0;$$

$1 \leq i \leq m$*;* $1 \leq p_0 < p_1 < p_2 < p_3 \leq m$*. Here,* $\hat{x}$ *means* x *is removed from the expression.*

For SL_2-matrices, we get:

COROLLARY 17. *Consider* $\mathbb{C}[SL_2 \oplus ... \oplus SL_2]^{SL_2}$*, where* SL_2 *acts by simultaneous conjugation on* m *matrices. Then*

(i) this algebra of invariants is generated by $Tr x_j$*,* $1 \leq j \leq m$*,* $Tr x_j x_k$*,*$1 \leq j < k \leq m$*, and*

$$s_3(x_{j_1}, x_{j_2}, x_{j_3}) := Tr(x_{j_1}, x_{j_2}, x_{j_3} - x_{j_1}, x_{j_3}, x_{j_2});\ 1 \leq j_1 < j_2 < j_3 \leq m;$$

(ii) the defining relations for the invariant ring are

$$Trs_3(x_{i_1}x_{i_2}x_{i_3}) \cdot Trs_3(x_{j_1}x_{j_2}x_{j_3}) + \\ 2\begin{vmatrix} Trx_{i_1}x_{j_1} - \frac{1}{2}Trx_{i_1}Trx_{j_1} & \cdots & Trx_{i_1}x_{j_3} - \frac{1}{2}Trx_{i_1}Trx_{j_3} \\ \vdots & \ddots & \vdots \\ Trx_{i_3}x_{j_1} - \frac{1}{2}Trx_{i_3}Trx_{j_1} & \cdots & Trx_{i_3}x_{j_3} - \frac{1}{2}Trx_{i_3}Trx_{j_3} \end{vmatrix} = 0;$$

$1 \leq i_1 < i_2 < i_3 \leq m$*;* $1 \leq j_1 < j_2 < j_3 \leq m$*; and*

$$\sum_{k=0}^{3}(-1)^k\left(Trx_ix_{p_k} - \frac{1}{2}Trx_i\,Trx_{p_k}\right)Trs_3(y_{p_0}, ..., \widehat{y_{p_k}}, ..., y_{p_3}) = 0;$$

$1 \leq i \leq m$*;* $1 \leq p_0 < p_1 < p_2 < p_3 \leq m$*; here,* Trx_i^2 *should be read as* $2 \cdot \big((Trx_i)^2 - 1\big)$*, and* $\hat{x}$ *means* x *is removed from the expression.*

PROOF. Proposition 6 tells us that, letting $(I') = (\det y_1 + \frac{1}{4}t_1^2 - 1, ..., \det y_m + \frac{1}{4}t_m^2 - 1)$, we have:

$$\mathbb{C}[SL_2^m]^{SL_2} \cong \big(\mathbb{C}[M_0^m]^{SL_2} \otimes \mathbb{C}[t_1] \otimes ... \otimes \mathbb{C}[t_m]\big)/(I').$$

Note that in this right coordinate ring, we have

$$\mathrm{Tr} y_i^2 = -2\det y_i = \frac{1}{2}t_i^2 - 2.$$

We can now eliminate the generators $\mathrm{Tr} y_i 2$ for $\mathbb{C}[M_0^m]^{SL_2}$ with the ideal (I'), and usi the isomorphism (1.1.3) and Lemma 4 to translate the results to $\mathbb{C}[SL_2^m]^{SL_2}$. Doing this gives the Corollary as stated. ☐

For the rest of this text we will usually consider the case of 3 matrices to calculate with. This is how the Corollary looks like in this case:

COROLLARY 18. *Consider* $\mathbb{C}[SL_2 \times SL_2 \times SL_2]^{SL_2}$, *where* SL_2 *acts by simultaneous conjugation on* 3 *matrices. Then:*

(i) this algebra of invariants is generated by Trx_1, Trx_2, Trx_3, Trx_1x_2, Trx_1x_3, Trx_2x_3, $Trx_1x_2x_3 - Trx_1x_3x_2$;

(ii) the only defining relation of the ring with respect to the above generator is:

$$\begin{aligned} 0 &= (Trx_1x_2x_3 - Trx_1x_3x_2)^2 + \\ & 2\begin{vmatrix} \frac{1}{2}(Trx_1)^2 - 2 & Trx_1x_2 - \frac{1}{2}Trx_1Trx_2 & Trx_1x_3 - \frac{1}{2}Trx_1Trx_3 \\ Trx_1x_2 - \frac{1}{2}Trx_1Trx_2 & \frac{1}{2}(Trx_2)^2 - 2 & Trx_2x_3 - \frac{1}{2}Trx_2Trx_3 \\ Trx_1x_3 - \frac{1}{2}Trx_1Trx_3 & Trx_2x_3 - \frac{1}{2}Trx_2Trx_3 & \frac{1}{2}(Trx_3)^2 - 2 \end{vmatrix}. \end{aligned}$$

More concretely, letting $a = Trx_1$, $b = Trx_2$, $c = Trx_3$, $d = Trx_1x_2$, $e = Trx_1x_3$, $f = Trx_2x_3$, $g = Trx_1x_2x_3 - Trx_1x_3x_2$, *the algebra of invariants can be seen as the following algebraic structure:*

$$\mathbb{C}[a,b,c,d,e,f,g]/(rel),$$

where

$$\begin{aligned} rel &= g^2 + 4a^2 + 4b^2 + 4c^2 + 4d^2 + 4e^2 + 4f^2 \\ &\quad -4ace - 4abd - 4bcf + 4def \\ &\quad -b^2e^2 - a^2f^2 - c^2d^2 \\ &\quad +2a^2bcf + 2abc^2d + 2ab^2ce \\ &\quad -2bcde - 2acdf - 2abef \\ &\quad -a^2b^2c^2 - 16. \end{aligned} \tag{1.3.3}$$

CHAPTER 2

Embedding into projective space

Instead of looking at the algebra of invariants of the the affine varieties $\mathbb{C}[SL_2^m]$, we can also embed this space into some projective variety, which enables us to look at this structure geometrically. In this chapter, we do this by direct computation: we define an embedding of $\mathbb{C}[SL_2^m]$ into a projective space, and construct a cover for most of the projective space of which we can calculate the invariants.

2.1. An embedding

In this chapter, we will focus on one such embedding. Letting

$$Q = \{(a : b : c : d : \Delta) \subset \mathbb{P}^4 \,|\, ad - bc = \Delta^2\},$$

our embedding is:

$$\begin{array}{rcl} SL_2^m & \overset{\phi}{\hookrightarrow} & Q \times ... \times Q \\ \left(\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), ..., \left(\begin{smallmatrix} a_m & b_m \\ c_m & d_m \end{smallmatrix}\right)\right) & \mapsto & \left((a_1 : b_1 : c_1 d_1 : 1), ..., (a_m : b_m : c_m : d_m : 1)\right). \end{array}$$

Thus, in effect we add the square roots of the determinants of the SL_2 matrices as extra coordinates to obtain something projective. The SL_2-action on SL_2^m then naturally extends to $Q \times ... \times Q$ by simultaneously conjugating the matrices formed by a_i, b_i, c_i, d_i and leaving the Δ_i intact. $Q \times ... \times Q$ is naturally a projective variety via the embedding in $\mathbb{P}^{5^m-1}$ with basis consisting of m-fold products of $a_i, b_i, c_i, d_i, \Delta_i$'s for $i = 1, ..., k$; for example, $Q \times Q$ embeds into $\mathbb{P}^{24}$:

$$\begin{array}{rcl} Q \times Q & \hookrightarrow & \mathbb{P}^{24} \\ ((a_1 : b_1 : c_1 : d_1 : \Delta_1), (a_2 : b_2 : c_2 : d_2 : \Delta_2)) & \mapsto & (a_1 a_2 : a_1 b_2 : a_1 c_2 : a_1 d_2 : \\ & & a_1 \Delta_2 : ... : \Delta_1 a_2 : \Delta_1 b_2 : \\ & & \Delta_1 c_2 : \Delta_1 d_2 : \Delta_1 \Delta_2). \end{array}$$

Our SL_2-action then naturally extends to a SL_2-action on $\mathbb{P}^{5^m-1}$.

The embedding of $X = SL_2^m$ is a so-called *compactification*: an embedding X of as a dense subset of a compact space.

Note that our choice of embedding is purely heuristic: there are many other possible projective spaces in which SL_2^m could be embedded. We chose this one because it seems intuitive: it preserves the separate matrices and the new coordinates we add have a concrete interpretation as the square root of the determinant. Another obvious choice would be to embed SL_2^m into a $\mathbb{P}^{4m}$ with a map:

$$(a_1, b_1, c_1, d_1, ..., d_m) \mapsto (a_1 : b_1 : ... : d_m : 1).$$

We will not study alternative embeddings in this text.

2.2. Covering $Q \times ... \times Q$

In topology, a *cover* of a set X is an indexed family of sets

$$U_A = \{U_\alpha \,|\, \alpha \in A\}$$

such that

$$X \subseteq \bigcup_{\alpha \in A} U_\alpha;$$

an *open cover* is a cover of which all U_α are open (see [**12**]).

In the Zariski topology, $\phi(SL_2^m)$ is obviously a subset of $Q \times ... \times Q$ that is open, affine, dense and SL_2-stable. To get more grip on $Q \times ... \times Q$, we will construct a cover of most of $Q \times ... \times Q$ by subsets sharing these properties.

Each of these subsets is then an affine space, so we can calculate its coordinate ring $\mathbb{C}[U_\alpha]$ and algebra of invariants $\mathbb{C}[U_\alpha]^{SL_2}$. Now, given U_α and U_β , the coordinate ring $\mathbb{C}[U_\alpha \cup U_\beta]$ can be found by looking at functions that are in the one coordinate ring and but not in the other, and then adding them.

Thus, $\mathbb{C}[U_\alpha \cup U_\beta]$ is described as an isomorphism between exensions of the coordinate rings of U_α and U_β (the so-called *glueing together* of the coordinate rings). This construction then of course also gives rise to an isomorphism between extensions of $\mathbb{C}[U_\beta]^{SL_2}$ and $\mathbb{C}[U_\beta]^{SL_2}$ to give $\mathbb{C}[U_\alpha \cup U_\beta]^{SL_2}$. Thus, by covering an $X \subset Q \times ... \times Q$ as large as we can, we can determine the algebra of invariants of a large part of $Q \times ... \times Q$.

Making such a cover is the main task we set for ourselves in this chapter. We will carry out the concrete calculations for the cases $m = 1$ and $m = 3$; especially the $m = 3$ case should give a good idea of how the situation for general m looks like.

2.3. The case $m = 1$

In the case $m = 1$, the embedding looks like this. Letting $A_0 = SL_2$, we have:

$$\begin{array}{ccl} A_0 & \overset{\phi}{\hookrightarrow} & Q = \{(a : b : c : d : \Delta) \subset \mathbb{P}^4 \mid ad - bc = \Delta^2\}, \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) & \mapsto & (a : b : c : d : 1)\,. \end{array}$$

As mentioned,

$$U_0 = \phi(A_0) = \{(a : b : c : d : \Delta) \in Q \mid \Delta \neq 0\} \tag{2.3.1}$$

is an open, affine, dense, SL_2-stable subset of Q, and since this is the case that we have been studying all along, we know what happens here: the coordinate ring of A_0 is

$$\mathbb{C}[a', b', c', d']/(a'd' - b'c' = 1),$$

and letting $y = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$, its ring of invariants is simply $\mathbb{C}[\mathrm{Tr} y]$.

We obtain another obvious subset of Q with the same properties by looking at

$$U_1 = \{(a'' : b'' : c'' : d'' : \Delta'') \in Q \mid a'' + d'' \neq 0\}. \tag{2.3.2}$$

To get a coordinate ring for A_1, we can assume $a'' + d'' = 1$ to obtain the embedding of the affine set

$$A_1 = \left\{ \left(\begin{pmatrix} a'' & b'' \\ c'' & 1 - a'' \end{pmatrix}, z' \right) \mid z''^2 = a'' - a''^2 - b''c'' \right\}$$

into Q. This space has coordinate ring $\mathbb{C}[a'', b'', c'', z'']/(a'' - a''^2 - b''c'' = z''^2)$, and SL_2 acts by conjugating the matrix and leaving the z'' intact. To obtain the ring of invariants, we proceed in the usual way by constructing an isomorphism of SL_2-representations to $M_0 \oplus \mathbb{C}$. One checks that this is done by

$$\begin{array}{ccc} A_1 & \overset{\psi}{\to} & M_0 \oplus \mathbb{C} \\ \left(\begin{pmatrix} a'' & b'' \\ c'' & 1 - a'' \end{pmatrix}, z'' \right) & \mapsto & \left(\begin{pmatrix} a'' - \frac{1}{2} & b'' \\ c'' & \frac{1}{2} - a'' \end{pmatrix}, z'' \right), \end{array}$$

and that the coordinate ring of $\psi(A_1)$ is

$$\mathbb{C}[a,b,c,t]/(-a^2-bc+\frac{1}{4}=t^2) \cong \mathbb{C}[a,b,c]\otimes \operatorname{span}\{1,t\}.$$

The right-hand side has ring of invariants $\mathbb{C}[\mathrm{Tr}x^2]\otimes \operatorname{span}\{1,t\}$, translating to $\mathbb{C}[t]$ for the left-hand side. Finally, this gives algebra of invariants $\mathbb{C}[A_1]^{SL_2} = \mathbb{C}[z'']$.

Thus, we now have two affine spaces A_0 and A_1 that can be embedded into Q. We can now glue them together. This is done simply by tracking the embeddings of A_0 and A_1 into Q adding coordinates to the coordinate rings as needed. We obtain the following two isomorphic coordinate rings for $\mathbb{C}[A_0 \cup A_1]$:

$$\begin{aligned}
\mathbb{C}[a',b',c',d',\frac{1}{a'+d'}] &\cong \mathbb{C}[a'',b'',c'',z'',\frac{1}{z''}]\\
a',b',c',d' &\mapsto \frac{a''}{z''},\frac{b''}{z''},\frac{c''}{z''},\frac{1-a''}{z''}\\
\frac{1}{a'+d'} &\mapsto z''\\
a'',b'',c'' &\mapsto \frac{a'}{a'+d'},\frac{b'}{a'+d'},\frac{c'}{a'+d'}\\
\tilde{z_1} &\mapsto \frac{1}{a'+d'}.
\end{aligned}$$

But then for the algebra of invariants, we get:

$$\begin{aligned}
\mathbb{C}[\mathrm{Tr}y,\frac{1}{\mathrm{Tr}y}] &\cong \mathbb{C}[z'',\frac{1}{z''}]\\
\mathrm{Tr}y &\leftrightarrow \frac{1}{z''}.
\end{aligned}$$

Note that the coordinate ring we constructed is exactly the coordinate ring of a $\mathbb{P}^1$.

Letting $A = \mathbb{B}$, we now have a cover U_A. We note that the points that are not in this cover satisfy $a+d=\Delta=0$. But then, up to conjugation, the only point in Q that is not in this cover is $n = (0:1:0:0:0)$ corresponding to a nilpotent matrix.

Now consider the embedding of $\mathbb{P}^4$ in $\mathbb{A}^5$ and consider the set of points

$$(0,\alpha,0,0,0);\ \alpha\neq 0$$

corresponding to n: the "lifts" of n. Suppose they belong to some closed set $V(S)\subset \mathbb{A}^5$, then we note that also $(0,0,0,0,0)\in V(S)$: for restricted to

$$S = \{(0,\alpha,0,0,0)\,|\,\alpha\in\mathbb{C}\}$$

any closed set given by a non-trivial polynomial can have only finitely many zeros, so in fact the polynomial defining the closed set must be constant restricted to S. We will see later that in geometric invariant theory, it is exactly these points with 0 in the closure of the lifts that will be ignored in the construction of a geometric quotient.

We summarize our results on covering Q in the following proposition:

PROPOSITION 19. *Letting $A = \mathbb{B}$, we have a cover U_A of*

$$\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} : \Delta\in Q \,|\, \begin{pmatrix} a & b\\ c & d\end{pmatrix} \text{ is not nilpotent}\right\}$$

given by (2.3.1) and (2.3.2). The algebra of invariants of this cover is isomorphic to $\mathbb{P}^1$.

2.4. The case $m = 3$

In this case, we have an embedding

$$\begin{array}{rcl} SL_2^3 & \overset{\phi}{\hookrightarrow} & Q \times Q \times Q \\ \left(\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_3 & b_3 \\ c_3 & d_3 \end{smallmatrix}\right)\right) & \mapsto & ((a_1 : b_1 : c_1 d_1 : 1), (a_2 : b_2 : c_2 : d_2 : 1), \\ & & ((a_3 : b_3 : c_3 : d_3 : 1)) \,. \end{array}$$

We can generalize the ideas used in the $m = 1$ case to construct a cover for $Q \times Q \times Q$. In this case, we can choose for each of the three matrices in the tuple independently whether we choose the trace or determinant to be non-zero.

Let us formalize this. Let $A = \mathbb{B}^3$, $f_{i,0} = \Delta_i$, $f_{i,1} = a_i + d_i$. Then our cover is

$$\bigcup_{a \in \mathbb{B}^3} U_a,$$

where

$$\begin{array}{rcl} U_a & = & \{(a_1 : b_1 : c_1 : d_1 : \Delta_1), ..., (a_m : b_m : c_m : d_m : \Delta_m) \\ & & \in Q \times Q \times Q \mid f_{1,a_1} \cdot f_{2,a_2} \cdot f_{3,a_3} \neq 0\}. \end{array} \tag{2.4.1}$$

Clearly, this covers exactly all the points in $Q \times Q \times Q$ corresponding to tuples of matrix, none of which are nilpotent.

As most of the calculations done in the $m = 3$ case are very similar to the $m = 1$ case, we will be pretty brief in describing them.

2.4.1. Glueing together $U_{(0,0,0)}$ and $U_{(1,0,0)}$.

We now glue together two of the affine parts of our covering: $U_{(0,0,0)}$, corresponding to our usual space $SL_2 \times SL_2 \times SL_2$, and $U_{(1,0,0)}$. Letting M_1 denote the affine space

$$M_1 = \left\{ \left(\left(\begin{array}{cc} a'' & b'' \\ c'' & 1 - a'' \end{array} \right), z' \right) \mid z''^2 = a'' - a''^2 - b''c'' \right\},$$

$U_{(1,0,0)}$ then corresponds to the space $M_1 \times SL_2 \times SL_2$.

$U_{(0,0,0)}$ clearly has coordinate ring

$$\begin{array}{rcl} \mathbb{C}[U_{(0,0,0)}] & = & \mathbb{C}[a'_1, b'_1, c'_1, d'_1, a'_2, b'_2, c'_2, d'_2, a'_3, b'_3, c'_3, d'_3] / \\ & & (a'_1 d'_1 - b'_1 c'_1 = 1, a'_2 d'_2 - b'_2 c'_2 = 1, a'_3 d'_3 - b'_3 c'_3 = 1) : \end{array}$$

the usual affine space that we are studying. Introducing generic matrices

$$y_i = \left(\begin{array}{cc} a'_1 & b'_1 \\ c'_1 & d'_1 \end{array} \right),$$

the action of SL_2 simply simultaneously conjugates the matrices y_i, and by Corollary 18 we know the algebra of invariants $\mathbb{C}[U_{(0,0,0)}]^{SL_2}$: this is

$$\mathbb{C}[\mathrm{Tr} y_1, \mathrm{Tr} y_2, \mathrm{Tr} y_3, \mathrm{Tr} y_1 y_2, \mathrm{Tr} y_1 y_3, \mathrm{Tr} y_2 y_3, \mathrm{Tr}(y_1 y_2 y_3 - y_1 y_3 y_2)]/(f),$$

where f is the relation (1.3.3) found earlier.

For $U_{(1,0,0)}$, in the same way as in the $m = 1$ case, we get a coordinate ring

$$\begin{array}{rcl} \mathbb{C}[U_{(1,0,0)}] & = & \mathbb{C}[a''_1, b''_1, c''_1, z''_1, a''_2, b''_2, c''_2, d''_2, a''_3, b''_3, c''_3, d''_3] / \\ & & (a''_1 - a''^2_1 - b''_1 c''_1 = z''^2_1, a'_2 d'_2 - b'_2 c'_2 = 1, a'_3 d'_3 - b'_3 c'_3 = 1), \end{array}$$

where SL_2 acts by conjugating the generic matrices

$$y''_1 = \left(\begin{array}{cc} a''_1 & b''_1 \\ c''_1 & 1 - a''_1 \end{array} \right), \; y''_2 = \left(\begin{array}{cc} a''_2 & b''_2 \\ c''_2 & d''_2 \end{array} \right), \; y''_3 = \left(\begin{array}{cc} a''_3 & b''_3 \\ c''_3 & d''_3 \end{array} \right),$$

and leaving z''_1 intact. The algebra of invariants $\mathbb{C}[U_{(1,0,0)}]^{SL_2}$ is

$$\mathbb{C}[z''_1, \mathrm{Tr} y''_2, \mathrm{Tr} y''_3, \mathrm{Tr} y''_1 y''_2, \mathrm{Tr} y_1 y''_3, \mathrm{Tr} y''_2 y''_3, \mathrm{Tr}(y''_1 y''_2 y''_3 - y''_1 y''_3 y''_2)]/(f''),$$

where f'' is a relation obtained in the same way as (1.3.3).

We now calculate $\mathbb{C}[U_{(0,0,0)} \cup U_{(1,0,0)}]$: this is

$$\begin{array}{rcl}
\mathbb{C}[a_1', b_1', c_1', d_1', a_2', ..., d_3', \frac{1}{a_1' + d_1'}]/(I') & \cong & \mathbb{C}[a_1'', b_1'', c_1'', z_1'', a_2'', ..., d_3'', \frac{1}{z_1''}]/(I'') \\
a_2', ..., d_3' & \leftrightarrow & a_2'', ..., d_3'' \\
a_1', b_1', c_1', d_1' & \mapsto & \frac{a_1''}{z_1''}, \frac{b_1''}{z_1''}, \frac{c_1''}{z_1''}, \frac{1 - a_1''}{z_1''} \\
\frac{1}{a_1' + d_1'} & \leftrightarrow & z_1'' \\
a_1'', b_1'', c_1'' & \mapsto & \frac{a_1'}{a_1' + d_1'}, \frac{b_1'}{a_1' + d_1'}, \frac{c_1'}{a_1' + d_1'}.
\end{array}$$

This gives rise to the following algebra of invariants $\mathbb{C}[U_{(0,0,0)} \cup U_{(1,0,0)}]^{SL_2}$:

$$\begin{array}{rcl}
\mathbb{C}[\mathrm{Tr}y_1, ..., \frac{1}{\mathrm{Tr}y_1}]/(...) & \cong & \mathbb{C}[z_1'', ..., \frac{1}{z_1''}]/(...) \\
\mathrm{Tr}y_1 & \leftrightarrow & \frac{1}{z_1''} \\
\mathrm{Tr}y_2, \mathrm{Tr}y_3 & \leftrightarrow & \mathrm{Tr}y_2'', \mathrm{Tr}y_3'' \\
\mathrm{Tr}y_1y_2, \mathrm{Tr}y_1y_3, \mathrm{Tr}y_2y_3 & \mapsto & \mathrm{Tr}y_1''y_2'' \cdot \frac{1}{z_1''}, \mathrm{Tr}y_1y_3'' \cdot \frac{1}{z_1''}, \mathrm{Tr}y_2''y_3'' \\
\mathrm{Tr}(y_1y_2y_3 - y_1y_3y_2) & \mapsto & \mathrm{Tr}(y_1''y_2''y_3'' - y_1''y_3''y_2'') \cdot \frac{1}{z_1''} \\
\mathrm{Tr}y_1''y_2'', \mathrm{Tr}y_1y_3'', \mathrm{Tr}y_2''y_3'' & \mapsto & \mathrm{Tr}y_1y_2 \cdot \frac{1}{\mathrm{Tr}y_1}, \mathrm{Tr}y_1y_3 \cdot \frac{1}{\mathrm{Tr}y_3}, \mathrm{Tr}y_2y_3 \\
\mathrm{Tr}(y_1''y_2''y_3'' - y_1''y_3''y_2'') \cdot \frac{1}{z_1''} & \mapsto & \mathrm{Tr}(y_1y_2y_3 - y_1y_3y_2) \cdot \frac{1}{\mathrm{Tr}y_1}
\end{array}$$

Here, (...) stands for the obvious set of relations: the usual variant of (1.3.3) and the relations $\mathrm{Tr}y_1 \cdot \frac{1}{\mathrm{Tr}y_1} = 1$ etcetera.

2.4.2. Glueing together $U_{(0,0,0)}$ and $U_{(1,1,1)}$. As a second and last example before attempting to tackle the total cover, we glue together $U_{(0,0,0)}$ and $U_{(1,1,1)}$. We claim outright that we get coordinate ring

$$\begin{array}{rcl}
\mathbb{C}[U_{(1,1,1)}] & = & \mathbb{C}[a_1'', b_1'', c_1'', z_1'', a_2'', b_2'', c_2'', z_2'', a_3'', b_3'', c_3'', z_3'']/ \\
& & (a_1'' - a_1''^2 - b_1''c_1'' = z_1''^2, a_2'' - a_2''^2 - b_2''c_2'' = z_2''^2, \\
& & a_3'' - a_3''^2 - b_3''c_3'' = z_3''^2),
\end{array}$$

with SL_2 acting by conjugating the generic matrices

$$y_1'' = \begin{pmatrix} a_1'' & b_1'' \\ c_1'' & 1 - a_1'' \end{pmatrix}, \; y_1'' = \begin{pmatrix} a_2'' & b_2'' \\ c_2'' & 1 - a_2'' \end{pmatrix}, \; y_3'' = \begin{pmatrix} a_3'' & b_3'' \\ c_3'' & 1 - a_3'' \end{pmatrix}$$

and leaving the z_i'' intact. The algebra of invariants $\mathbb{C}[U_{(1,1,1)}]^{SL_2}$ is, as usual,

$$\mathbb{C}[z_1'', z_2'', z_3'', \mathrm{Tr}y_1''y_2'', \mathrm{Tr}y_1y_3'', \mathrm{Tr}y_2''y_3'', \mathrm{Tr}(y_1''y_2''y_3'' - y_1''y_3''y_2'')]/(f'')$$

for some variant f'' of (1.3.3).

The coordinate ring $\mathbb{C}[U_{(0,0,0)} \cup U_{(1,1,1)}]$ is as follows:

$$\begin{array}{rcl}
\mathbb{C}[a'_1, ..., d'_3, \frac{1}{a'_1 + d'_1}, \frac{1}{a'_2 + d'_2} \frac{1}{a'_3 + d'_3}]/(I') & \cong & \mathbb{C}[a''_1, ..., d''_3, \frac{1}{z''_1}, \frac{1}{z''_2}, \frac{1}{z''_3}]/(I'') \\
a'_i, b'_i, c'_i, d'_i & \mapsto & \frac{a''_i}{z''_i}, \frac{b''_i}{z''_i}, \frac{c''_i}{z''_i}, \frac{1 - a''_i}{z''_i} \\
\frac{1}{a'_i + d'_i} & \leftrightarrow & z''_i \\
a''_i, b''_i, c''_i & \mapsto & \frac{a'_i}{a'_i + d'_i}, \frac{b'_i}{a'_i + d'_i}, \frac{c'_i}{a'_i + d'_i}.
\end{array}$$

The two isomorphic forms of $\mathbb{C}[U_{(0,0,0)} \cup U_{(1,1,1)}]^{SL_2}$ this gives rise to are:

$$\begin{array}{rcl}
\mathbb{C}[\mathrm{Tr}y_1, ..., \frac{1}{\mathrm{Tr}y_1}, \frac{1}{\mathrm{Tr}y_2}, \frac{1}{\mathrm{Tr}y_3}]/(...) & \cong & \mathbb{C}[z''_1, ..., \frac{1}{z''_1}, \frac{1}{z''_2}, \frac{1}{z''_3}]/(...) \\
\mathrm{Tr}y_i & \leftrightarrow & \frac{1}{z''_i} \\
\mathrm{Tr}y_i y_j & \mapsto & \mathrm{Tr}y''_i y''_j \cdot \frac{1}{z''_i} \cdot \frac{1}{z''_j} \\
\mathrm{Tr}(y_1 y_2 y_3 - y_1 y_3 y_2) & \mapsto & \mathrm{Tr}(y''_1 y''_2 y''_3 - y''_1 y''_3 y''_2) \cdot \frac{1}{z''_1} \cdot \frac{1}{z''_2} \cdot \frac{1}{z''_3} \\
\mathrm{Tr}y''_i y''_j & \mapsto & \mathrm{Tr}y_i y_j \cdot \frac{1}{\mathrm{Tr}y_i} \cdot \frac{1}{\mathrm{Tr}y_j}, \\
\mathrm{Tr}(y''_1 y''_2 y''_3 - y''_1 y''_3 y''_2) & \mapsto & \mathrm{Tr}(y_1 y_2 y_3 - y_1 y_3 y_2) \cdot \frac{1}{\mathrm{Tr}y_1} \cdot \frac{1}{\mathrm{Tr}y_2} \cdot \frac{1}{\mathrm{Tr}y_3}.
\end{array}$$

2.4.3. The complete cover. Now let us see what happens if, for instance, we want to determine the coordinate ring of

$$\mathbb{C}[U_{(0,0,0)} \cup U_{(1,1,1)} \cup U_{(1,0,0)}].$$

We do this by glueing together the coordinate rings

$$\mathbb{C}[\mathrm{Tr}y_1, ..., \frac{1}{\mathrm{Tr}y_1}, \frac{1}{\mathrm{Tr}y_2}, \frac{1}{\mathrm{Tr}y_3}]/(...),$$

which is the coordinate ring $\mathbb{C}[U_{(0,0,0)} \cup U_{(1,1,1)}]$ seen as an extension of $\mathbb{C}[U_{(0,0,0)}]$, and the coordinate ring

$$\begin{array}{rcl}
\mathbb{C}[U_{(1,0,0)}] & = & \mathbb{C}[a''_1, b''_1, c''_1, z''_1, a''_2, b''_2, c''_2, d''_2, a''_3, b''_3, c''_3, d''_3]/ \\
 & & (a''_1 - a''^2_1 - b''_1 c''_1 = z''^2_1, a'_2 d'_2 - b'_2 c'_2 = 1, a'_3 d'_3 - b'_3 c'_3 = 1).
\end{array}$$

But then to glue these rings together, it suffices to add the new coordinates

$$\frac{1}{a''_2 + d''_2}, \frac{1}{a''_3 + d''_3}$$

to the coordinate ring of $\mathbb{C}[U_{(0,0,0)}]$, and the coordinate ring of $\mathbb{C}[U_{(0,0,0)} \cup U_{(1,1,1)}]$ remains unchanged. We summarize this result in the following claim:

PROPOSITION 20. *The cover* $\bigcup_{a \in \mathbb{B}^3} U_a$ *with* U_a *defined in (2.4.1) is a cover by open, affine, dense and* SL_2*-stable subsets of* $Q \times Q \times Q$*. It covers exactly the points in* $Q \times Q \times Q$ *corresponding to triples of non-nilpotent matrices. Its coordinate ring is*

$$\mathbb{C}[a'_1, b'_1, c'_1, d'_1, a'_2, b'_2, c'_2, d'_2, a'_3, b'_3, c'_3, d'_3, \frac{1}{a'_1 + d'_1}, \frac{1}{a'_2 + d'_2} \frac{1}{a'_3 + d'_3}];$$

its algebra of invariants is

$$\begin{aligned}&\mathbb{C}[\mathit{Tr}y_1,\, \mathit{Tr}y_2,\, \mathit{Tr}y_3,\, \mathit{Tr}y_1y_2,\, \mathit{Tr}y_1y_3,\, \mathit{Tr}y_2y_3,\, \mathit{Tr}(y_1y_2y_3 - y_1y_3y_2),\\ &\frac{1}{\mathit{Tr}y_1}, \frac{1}{\mathit{Tr}y_2}, \frac{1}{\mathit{Tr}y_3}]/(I),\end{aligned}$$

where

$$(I) = \left(f,\, \mathit{Tr}y_1 \cdot \frac{1}{\mathit{Tr}y_1} = 1,\, \mathit{Tr}y_2 \cdot \frac{1}{\mathit{Tr}y_2} = 1,\, \mathit{Tr}y_3 \cdot \frac{1}{\mathit{Tr}y_3} = 1\right);$$

f as in (1.3.3).

Note that there are more affine parts of $Q \times Q \times Q$ that our cover does not reach; for instance, we could define an affine part

$$U = \{(M_1 : \Delta_1, M_2 : \Delta_2, M_3 : \Delta_3) \in Q \times Q \times Q \,|\, \mathrm{Tr} M_1 M_2 \mathrm{Tr} M_3 \neq 0\}$$

to which such triples as

$$\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : 0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : 0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : 1\right) \in Q \times Q \times Q$$

belong. However, in the case of the affine space U it is not so clear how to calculate the algebra of invariants looks like: our usual construction of a module isomorphism M_0 does not seem to work in this case. In the next chapter on geometric invariant theory we will study in more detail the general conditions under which affine parts exist.

CHAPTER 3

Geometry of the projective space

In the previous chapter, we introduced an embedding of $\mathbb{C}[SL_2^m]$ into a projective space, and constructed a cover of most of this space of which we were able to calculate the invariants. This approach is the underlying idea of the field of geometric invariant theory (GIT).

In this chapter we introduce some of the concepts of GIT and see how they apply to our situation. In sections 3.1 and 3.2, we introduce GIT, and in section 3.3 we see how we can look at geometric quotients in the case of a SL_2-action. In the following sections, we perform calculations for the cases $m = 1$, $m = 2$ and $m = 3$ and find some results for general m.

3.1. Quotients, stability

The goal of geometric invariant theory (GIT) is to create a so-called *geometric quotient.* Geometric invariant theory was developed by David Mumford in 1965, and his book [**8**] is the primary source about this field. The definitions we give below were taken mainly from [**6**] and [**7**, Ch.6]: two sources where the concepts of GIT are introduced in a more accessible way.

Definition 21. Let X be an algebraic variety with an action by a reductive group G. A pair (Y, ϕ) is called a *categorical quotient* if:
(i) Y is an algebraic variety;
(ii) ϕ is a morphism of varieties from X to Y;
(iii) ϕ is G-invariant;
(iv) ϕ has the *universal mapping property*: given any algebraic variety Z and a morphism $\psi : X \to Z$ which is constant on G-orbits, there exists a unique $\theta : Y \to Z$ such that $\psi = \theta \circ \phi$; that is, we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[dr, "\psi"'] & Y \arrow[d, "\theta"] \\
 & Z,
\end{tikzcd}$$

The important property is of course (iv): it indicates that the map separates the orbits of X under G as well as possible. If such a categorical quotient (Y, ϕ) exists, we denote it by $X//G$. We also have a *good quotient*, which satisfies some stronger conditions:

Definition 22. Let X be an algebraic variety with an action by a reductive group G. A pair (Y, ϕ) consisting of an algebraic morphism Y and a morphism ϕ is called a *good quotient* if it satisfies the following conditions:
(i) ϕ is surjective;
(ii) ϕ is G-invariant;
(iii) ϕ is an affine morphism (ie, the inverse image of an open affine set is again an open affine set);
(iv) if $W \subset X$ is G-stable and closed, then $\phi(W)$ is closed;

(v) let $X_1, X_2 \subset X$ disjoint, G-stable and closed, then $\phi(W_1) \cap \phi(W_2) = \emptyset$;
(vi) for open $U \subset Y$, $\phi^* : k[U] \to k[\phi^{-1}(U)]^G$ is an isomorphism.

One can show that a good quotient (Y, ϕ) is also a categorical quotient. If the set of points of Y is in bijection with the set of orbits in X, then (Y, ϕ) is called a *geometric quotient*. This latter statement is equivalent to all orbits of X under G being closed.

Now if X is an affine variety and G is a so-called "reductive group" (SL_2 is such a reductive group), then we know what the categorical quotient looks like. For this, we need the construction from algebraic geometry of the *spectrum* of a ring, denoted Spec$R$. This is a construction to create an affine variety having R as its coordinate ring.

So, if we have an algebra of invariants $k[W]^{SL_2}$, then Spec$k[W]^{SL_2}$ is an affine variety with coordinate ring $k[W]^{SL_2} \subset k[W]$, and this inclusion induces a map $\phi : W \to \mathrm{Spec}k[W]^{SL_2}$: given a point it simply gives the values of the invariant functions evaluated in that point. One can then show that ϕ is a good quotient for SL_2 acting on W.

It turns out that we can not, in general, construct a good quotient for a complete projective variety X: the best we can do is to construct a quotient on some open subvariety of X. This is where glueing, as we did in Chapter 2, comes in. For if we have a cover of some subset $X' \subset X$ by affine spaces:

$$X' = \bigcup_{\alpha \in A} U_\alpha,$$

then as indicated we can calculate its algebra of invariants X'^G by glueing together the algebras of invariants $k[U_\alpha]^{SL_2}$. But then as above, this induces maps $\phi_\alpha : U_\alpha \to \mathrm{Spec}k[U_\alpha]^{SL_2}$ that are good quotients of the U_α. Finally, one can check that the map $\phi : X' \to \mathrm{Spec}X'^G$ obtained by combining all ϕ_α is a good quotient for X'.

3.2. Semi-stable and stable points and the Hilbert-Mumford criterion

The question is then what points of a projective variety X can be covered by open, affine, G-stable sets. For this, GIT introduces the following definitions:

DEFINITION 23. Let V be a G-module and $X \hookrightarrow \mathbb{P}(V)$ be a projective variety so that G acts linearly on X. Let be $\hat{X} \hookrightarrow V$ the corresponding affine variety so that we have the canonical map

$$\hat{X} \backslash \{0\} \xrightarrow{\phi} X : x \overset{\phi}{\mapsto} [x].$$

A point $x \in X$ is then called *semi-stable* if there is a point $\hat{x} \in \hat{X}$; $\phi(\hat{x}) = x$ (a *lift* of x) so that $0 \notin \overline{G.\hat{x}}$. Similarly, x is called *stable* if there is an $\hat{x}$ so that $G \cdot \hat{x}$ is closed, and $\dim(G \cdot \hat{x}) = \dim G$. A point is called *unstable* if it is not semi-stable. (Note that unstable is *not* the opposite of stable!)

Denote $X^{ss} = \{x \in X \mid x \text{ is semi-stable}\}$ and $X^s = \{x \in X \mid x \text{ is stable}\}$.

Then the following holds:

THEOREM 24. *Let X be a projective variety. Then X^{ss} admits a good quotient ϕ. Also, $\phi : X^s \to \phi(X^s)$ is a geometric quotient.*

However, suppose we have a good quotient not for the whole of X^{ss}, but for some part of it. The following Corollary indicates how we can still find a geometric quotient:

PROPOSITION 25. *Let $X^s \subset X'$ and let (Y, ϕ) be a good quotient for X'. If $\phi(X^s)$ is open in Y, then $\phi : X^s \to \phi(X^s)$ is a geometric quotient.*

Proof. We use [**7**, Proposition 6.1.10]. This tells us the following: let $U \subset X'$ be the set of points $x \in X'$ such that the orbit of x is closed in X' and the stabilizer of x' is finite. Then $V = \phi(U)$ is open in Z, and $\phi|_U : U \to V$ is a geometric quotient. Note that $X^s \subset U$, so $\phi(X^s) \subset V$, and because $\phi(X^s)$ is open in Y, it is also open in V. Now by Property (ii) of [**6**, Lecture 6, section 3] applied to $\phi|_U$ and $\phi(X^s) \subset V$, $\phi : X^s \to \phi(X^s)$ is a geometric quotient. $\square$

In [**7**, Lemma 6.4.5] it is shown that in the projective case, a point $x \in X$ being semi-stable is equivalent to the existence of a homogeneous G-invariant polynomial with strictly positive degree which does not vanish at x. Note that this corresponds exactly to the existence of an open, affine, G-stable set containing x, which corresponds to our previous discussion on coverings.

To determine whether points in a projective variety are (semi-)stable, one can use the Hilbert-Mumford criterion. This criterion of formulated by looking at the action of so-called "one-parameter subgroups" of an algebraic group G; these are simply algebraic group morphisms $\lambda : \mathbb{C}^* \to G$. We suppose that G acts linearly on X.

Suppose we have a projective variety $X \subset \mathbb{P}(V)$, corresponding affine variety $\hat{X} \subset V$ and canonical map ϕ, as above. Let $x = \phi(\hat{x})$ and λ be given. In [**6**] it is shown that by choosing a basis $\{e_i\}$ for V suitably, we can write the action of λ on V as:

$$\lambda(t) \cdot \hat{x} = \sum c_i t^{r_i} e_i.$$

The *weights* of x with respect to λ are now defined to be the $\{r_i \,|\, c_i \neq 0\}$. We can now formulate the criterion as follows:

Theorem 26. *(Hilbert-Mumford) In the above setting, x is semi-stable if and only if for every λ, x has at least one non-positive weight; x is stable if and only if for every non-trivial λ, it has at least one negative weight.*

3.3. Our case

Let is now try to apply this theory to our normal projective situation in which we project every matrix in one $\mathbb{P}^4$. Let

$$Q = \left\{(a : b : c : d : \Delta) \,|\, ad - bc = \Delta^2\right\} \subset \mathbb{P}^4$$

be the embedding of one such matrix. We are then interested in finding out the quotient of a product of several Q's.

For this, we will need to look at one-parameter subgroups of SL_2. The following lemma helps:

Lemma 27. *Let $\lambda(t)$ be a non-trivial one-parameter subgroup of SL_2. Then there is a non-trivial one-parameter subgroup $\tilde{\lambda}$ and a similarity matrix Q so that $\lambda(t) = Q^{-1} diag(t, t^{-1}) Q$, and λ and $\tilde{\lambda}$ have the same stable and semi-stable points.*

Proof. Let us take a one-parameter subgroup $\lambda(t)$ of SL_2, which is by definition an algebraic group morphism $\lambda : \mathbb{C}^* \to SL_2$. Because λ is non-trivial and algebraic, Kerλ is an algebraic subgroup Ker$\lambda \subsetneq \mathbb{C}^*$, so it must be a finite group of nth roots of unity, the cyclic group C_n. By letting $\tilde{\lambda}(z) = \lambda(z^n)$, we see that then $\tilde{\lambda}$ is an isomorphism $\mathbb{C}^* \to \lambda(\mathbb{C}^*)$:

$$\begin{tikzcd}
\mathbb{C}^*/C_n \arrow[r, "\lambda"] \arrow[d, "z \mapsto z^n"'] & SL_2 \\
\mathbb{C}^*, \arrow[ur, dashed, "\tilde{\lambda}"'] &
\end{tikzcd}$$

Because clearly λ and $\tilde{\lambda}$ have the same stable and semi-stable points (for a given point, $\tilde{\lambda}$ has the same weights as λ multiplied by n), without loss of generality we may assume λ is an isomorphism $\mathbb{C}^* \to \lambda(\mathbb{C}^*)$.

Now suppose that some $A \in \lambda(\mathbb{C}^*)$ is not diagonalizable, then in Jordan form either $A \in \lambda(\mathbb{C}^*)$ or $-A \in \lambda(\mathbb{C}^*)$ can be written as

$$\hat{A} = Q^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Q$$

using some similarity matrix Q. But then also

$$\hat{A}^n = Q^{-1} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} Q \in \lambda(\mathbb{C}^*) \quad \forall n \in \mathbb{N}.$$

Note that this set is Zariski-dense in the affine variety

$$V = \left\{ Q^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} Q \,|\, x \in \mathbb{C} \right\},$$

because for any closed set $Z(f_1, ..., f_n) \subsetneq V$, each $f_n \in k[V]$ can have only finitely many values for x for which it is zero. Because $\lambda(\mathbb{C}^*)$ is closed as the image of a closed set, then certainly $V \subset \lambda(\mathbb{C}^*)$. But this means we have an infinite additive algebraic subgroup of $\mathbb{C}^*$, which is impossible.

So in fact, all matrices in $\lambda(\mathbb{C}^*)$ are diagonalizable, and because it is a commutative group, they are in fact simultaneously diagonalizable. For suppose v is an eigenvector of a matrix A so that $Av = \mu v$ for some V, then $A(Bv) = B(Av) = B(\lambda v) = \lambda Bv$, so $B(v)$ is an eigenvector of A corresponding to the same eigenvalue λ, so $Bv = \mu v$ for some μ. (This assumes that A does not have a double eigenvalue, but in this case it is $* \cdot \mathrm{Id}$ which is invariant under conjugation.)

By the same Zariski arguments, we then have that for some similarity matrix Q,

$$\lambda(\mathbb{C}^*) = \left\{ Q \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} Q^{-1} \,|\, x \neq 0 \right\} \cong \mathbb{C}^*.$$

Thus $\lambda(t) = Q^{-1}\mathrm{diag}(t, t^{-1})Q$ or $\lambda(t) = Q^{-1}\mathrm{diag}(t^{-1}, t)Q$ because we assumed λ to be an isomorphism $\mathbb{C}^* \to \lambda(\mathbb{C}^*)$. In the first case, we are done; in the second case, using another similarity matrix Q that whitches the two eigenvalues gives us the desired conclusion as well. ☐

We also introduce the following notion:

DEFINITION 28. (See [**2**, p.18]) A matrix tuple $(A_1, A_2, ..., A_n) \in M^k_{2\times 2}$ is called *reducible* if and only if it can be simultaneously conjugated to the form

$$\left(\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, ..., \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right);$$

it is called *irreducible* otherwise.

3.4. One matrix

First of all, let us look at the situation when we have just one matrix. The action on Q (conjugating the matrix formed by a, b, c, d and leaving Δ intact) extends to an action on the whole of $\mathbb{P}^4$, so we can consider its semi-stable points.

Let us apply the Hilbert-Mumford criterion to find unstable points. For a matrix M to be unstable, there must be a one-parameter subgroup such that all weights of M are positive. By Lemma 27 we can assume that M acts diagonally, so in some basis of Q, the action of SL_2 is:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \Delta \right) \mapsto \left(\begin{pmatrix} a & t^2 b \\ t^{-2} c & d \end{pmatrix}, \Delta \right).$$

Thus, we must have that $b \neq 0$, and this corresponds to the one unstable point $n \in Q$ as below:

$$v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \leftrightarrow n = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : 0 \right).$$

In other words, the unstable points are all in the one orbit of nilpotent matrices.

Theorem 24 now tells us that $Q^{ss} = Q \backslash \{\text{nilpotent matrices}\}$ admits a good quotient ϕ. Letting

$$A = \{(a, b, c, d, \Delta) \in \mathbb{A}^5 \,|\, ad - bc = 1\},$$

this quotient is induced by the inclusion $A^{SL_2} \subset A$. But, as we know, $A^{SL_2} = \mathbb{C}[a + d, \Delta]$, so the induced quotient map is simply:

$$\begin{array}{rcl} \phi : Q^{ss} & \rightarrow & \mathbb{P}^1 \\ (M : \Delta) & \mapsto & (\mathrm{tr} M : \Delta). \end{array}$$

Indeed, this quotient is not defined exactly when $\mathrm{Tr} M = \det M = 0$, ie when M is nilpotent.

We can also try to apply Hilbert-Mumford criterion to find stable points. We find that for a matrix M to be not stable, there has to be a λ such that all weights are non-negative. This means that conjugated suitably, we must have $M_{21} = 0$. But of course this holds for any matrix: just write it in Jordan form. This is not so strange when one considers the definition of points being stable: for this we need that the dimension (in affine space) of the orbit of the matrix equals $\dim(SL_2) = 3$. But conjugation leaves the trace and matrix of the matrix intact, so this can never be the case.

3.5. Two matrices

Now let us consider

$$\begin{array}{rcl} P & = & Q \times Q = (\left\{(a_1 : b_1 : c_1 : d_1 : \Delta_1) \,|\, a_1 d_1 - b_1 c_1 = \Delta_1^2\right\} \times \\ & & \left\{(a_2 : b_2 : c_2 : d_2 : \Delta_2) \,|\, a_2 d_2 - b_2 c_2 = \Delta_2^2\right\}. \end{array}$$

As indicated before, to see $Q \times Q$ as a projective space, we embed it as $P \subset \mathbb{P}^4 \times \mathbb{P}^4 \rightarrow \mathbb{P}^{24}$.

3.5.1. Constructing a good quotient by covers.

As indicated above, the construction of a good quotient of a subset of points of $Q \times Q$ is done by first calculating the algebras of invariants of affine parts, and then glueing together the induced good quotients on the affine parts. Let us take a cover by $\mathbb{B}^2$ of $(Q \times Q)^{nn}$ consisting of the points that correspond to non-nilpotent matrices like we did for three matrices in Section 2.4.

For example, we get an affine open set

$$U_{(0,0)} = \{((M_1 : \Delta_1), (M_2 : \Delta_2)) \,|\, \Delta_1 \Delta_2 \neq 0\}$$

which can be identified with SL_2^2 with coordinate ring $\mathbb{C}[M_1, M_2]$ and algebra of invariants $\mathbb{C}[\mathrm{Tr} M_1, \mathrm{Tr} M_2, \mathrm{Tr} M_1 M_2]$. As functions on $Q \times Q$ these generators are

$$\frac{\mathrm{Tr} M_1 \cdot \Delta_2}{\Delta_1 \cdot \Delta_2}, \ \frac{\Delta_1 \cdot \mathrm{Tr} M_2}{\Delta_1 \cdot \Delta_2}, \ \frac{\mathrm{Tr} M_1 M_2}{\Delta_1 \cdot \Delta_2},$$

so we have a good quotient $U_{(0,0)} \rightarrow \{(a : b : c : d) \in \mathbb{P}^3 \,|\, d \neq 0\}$:

$$((M_1 : \Delta_1), (M_2 : \Delta_2)) \mapsto (\mathrm{Tr} M_1 \Delta_2 : \Delta_1 \mathrm{Tr} M_2 : \mathrm{Tr} M_1 M_2 : \Delta_1 \Delta_2).$$

Similarly, for the other affine parts one finds good quotients to $\{(a : b : c : d) \in \mathbb{P}^3 \,|\, d \neq 0\}$ of $U_{(1,0)}$, $U_{(0,1)}$ and $U_{(1,1)}$ as follows:

$$\begin{array}{rcl} ((M_1 : \Delta_1), (M_2 : \Delta_2)) & \mapsto & (\mathrm{Tr}M_1\mathrm{Tr}M_2 : \mathrm{Tr}M_1M_2 : \Delta_1\Delta_2 : \mathrm{Tr}M_1\Delta_2); \\ ((M_1 : \Delta_1), (M_2 : \Delta_2)) & \mapsto & (\mathrm{Tr}M_1\mathrm{Tr}M_2 : \mathrm{Tr}M_1M_2 : \Delta_1\Delta_2 : \Delta_1\mathrm{Tr}M_2); \\ ((M_1 : \Delta_1), (M_2 : \Delta_2)) & \mapsto & (\Delta_1\mathrm{Tr}M_2 : \mathrm{Tr}M_1\Delta_2 : \mathrm{Tr}M_1M_2 : \mathrm{Tr}M_1\mathrm{Tr}M_2). \end{array}$$

But then let us consider the map

$$\begin{array}{rcl} (3.5.1)\, \phi : (Q \times Q)^{nn} & \to & \phi((Q \times Q)^{nn}) \subset \mathbb{P}^4 \\ (M_1 : \Delta_1, M_2 : \Delta_2) & \mapsto & (\mathrm{Tr}M_1\Delta_2 : \Delta_1\mathrm{Tr}M_2 : \mathrm{Tr}M_1\mathrm{Tr}M_2 : \Delta_1\Delta_2 : \mathrm{Tr}M_1M_2). \end{array}$$

Restricting this map to $U_{(0,0)}$, for instance, gives:

$$\phi : U_{(0,0)} \to \{(a : b : c : d : e) \in \mathbb{P}^4 \,|\, ab = cd \wedge d \neq 0\}.$$

But now note that via $\psi : (a : b : c : d : e) \mapsto (a : b : e : d)$ we have a commutative diagram

$$\begin{tikzcd} U_{(0,0)} \arrow[r] \arrow[d, "\phi"'] & \{(a : b : c : d) \in \mathbb{P}^3 \,|\, d \neq 0\} \\ \phi(U_{(0,0)}) \arrow[ru, dashed, "\cong", "\psi"'] & \end{tikzcd}$$

so ϕ is a good quotient for $U_{(0,0)}$. Similar constructions can be drawn for the other affine parts. Finally, we arrive at the conclusion:

Proposition 29. Let

$$Y = \left\{(x_1 : x_2 : x_3 : x_4 : x_5) \in \mathbb{P}^4 \,|\, x_1x_2 = x_3x_4\right\} \backslash (0 : 0 : 0 : 0 : 1),$$

ϕ as above. Then (Y, ϕ) is a good quotient for $(Q \times Q)^{nn}$.

Note that as in Chapter 2, $(Q \times Q)^{nn}$ does not cover all the semi-stable points: for example, we could define an affine subset

$$\{((M_1 : \Delta_1), (M_2 : \Delta_2)) \,|\, \mathrm{Tr}M_1M_2 \neq 0\}.$$

Also, we note another approach we might try to obtain the quotient map: this is by considering the algebra of invariants for the complete cover we calculated in 2, and trying to turn that into a quotient map. But for example, consider the algebra of invariants as an extension of the affine space where $\Delta_1 = \Delta_2 = 1$. Then as functions on $Q \times Q$ the generators are

$$\left\{\frac{\mathrm{Tr}M_1\Delta_2}{\Delta_1\Delta_2}, \frac{\Delta_1\mathrm{Tr}M_2}{\Delta_1\Delta_2}, \frac{\mathrm{Tr}M_1M_2}{\Delta_1\Delta_2}, \frac{\Delta_1\Delta_2}{\mathrm{Tr}M_1\Delta_2}, \frac{\Delta_1\Delta_2}{\Delta_1\mathrm{Tr}M_2}\right\},$$

and it is not so clear how to make a quotient map of this: for example, the obvious map

$$\begin{array}{rcl} ((M_1 : \Delta_1), (M_2, \Delta_2)) & \mapsto & ((\mathrm{Tr}M_1)^2\mathrm{Tr}M_2\Delta_2 : \mathrm{Tr}M_1(\mathrm{Tr}M_2)^2\Delta_1 : \\ & & \mathrm{Tr}M_1M_2\mathrm{Tr}M_1\mathrm{Tr}M_2 : \Delta_1^2\mathrm{Tr}M_2\Delta_2 : \Delta_1\mathrm{Tr}M_1\Delta_2^2 : \\ & & \Delta_1\Delta_2\mathrm{Tr}M_1\mathrm{Tr}M_2) \end{array}$$

is not even well-defined as it gives $(0 : 0 : 0 : 0 : 0 : 0)$ whenever any of the matrices is nilpotent. Thus, to construct a quotient, glueing together quotients seems a better approach then glueing together invariant rings.

3.5.2. Stability of points. We can now use Hilbert-Mumford to find the unstable, semi-stable and stable points in this projective space and find out in general under which conditions we can define good and geometric quotients.

LEMMA 30. *The unstable points of P embedded in $\mathbb{P}^{24}$ correspond to reducible pairs of matrices (A, B) of which at least one is nilpotent.*

PROOF. If we recall how a one-parameter subgroup $\mathrm{diag}(t, t^{-1})$ acts on one matrix:

$$\left(\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \Delta_i\right) \mapsto \left(\begin{pmatrix} a_i & t^2 b_i \\ t^{-2} c_i & d_i \end{pmatrix}, \Delta_i\right),$$

we can easily see how it acts on a basis element of $\mathbb{P}^{24}$ as well: for example, it acts on $b_1\Delta_2$ as $b_1\Delta_2 \mapsto t^2 b_1 \cdot \Delta_2 = t^2(b_1\Delta_2)$. By Hilbert-Mumford, a point is unstable if there is a one-parameter subgroup so that all weights are positive, so a point$p \in P$ if and only if (under some basis transformation):

$$a_1a_2 = a_1c_2 = a_1d_2 = a_1\Delta_2 = b_1c_2 = c_1a_2 = c_1b_2 = c_1c_2 = c_1d_2 = c_1\Delta_2 =$$
$$d_1a_2 = d_1c_2 = d_1d_2 = d_1\Delta_2 = \Delta_1a_2 = \Delta_1c_2 = \Delta_1d_2 = \Delta_1\Delta_2 = 0.$$

Suppose we have an unstable point, so the above relations hold. Since all combinationes of $\{a_1, b_1, c_1, \Delta_1\} \cdot \{a_2, b_2, c_2, \Delta_2\}$ occur, for some i we must have $a_i = b_i = c_i = \Delta_i = 0$; say this is true for $i = 1$. The only relation that still remains is $b_1c_2 = 0$, which exactly states that the matrix pair is reducible.

Conversely, if one matrix is nilpotent, say A, then we can write it in strictly upper triangular form, and the only remaining relation $b_1c_2 = 0$ now implies reducibility. □

COROLLARY 31. *In the above context, a matrix pair $(M_1, M_2) \in P$ is semi-stable if and only if it is irreducible or neither matrix is nilpotent.*

PROPOSITION 32. *In the above context, a point in P is stable if and only if M_1, M_2 is irreducible and neither matrix is nilpotent.*

PROOF. A point is not stable if and only if there is one-parameter subgroup λ so that all weights with respect to λ are ≥ 0. First suppose we have a matrix pair (A, B) that is not stable. This gives the following equations:

$$c_1a_2 = c_1c_2 = c_1d_2 = c_1\Delta_2 = a_1c_2 = d_1c_2 = \Delta_1c_2 = 0.$$

If $c_1 \neq 0$, then $a_2 = c_2 = d_2 = \Delta_2 = 0$, so matrix B is nilpotent, and similarly for matrix A. Otherwise, $c_1 = c_2 = 0$ and the pair is reducible. The converse clearly also holds. □

3.5.3. (Semi-)stability and quotients. In Proposition 29, we constructed a good quotient of $(Q \times Q)^{nn}$, the set of pairs where none of the matrices is nilpotent. In light of the previous calculations, we now see that

$$(Q \times Q)^{ss} \subsetneq (Q \times Q)^{nn} \subsetneq (Q \times Q)^{ss}.$$

Now, $(Q \times Q)^{nn}$ had quotient space

$$Y = \left\{(x_1 : x_2 : x_3 : x_4 : x_5) \in \mathbb{P}^4 \,|\, x_1x_2 = x_3x_4\right\} \backslash (0 : 0 : 0 : 0 : 1),$$

with the quotient map ϕ from (3.5.1):

$$(M_1 : \Delta_1, M_2 : \Delta_2) \mapsto (\mathrm{Tr}M_1\Delta_2 : \Delta_1\mathrm{Tr}M_2 : \mathrm{Tr}M_1\mathrm{Tr}M_2 : \Delta_1\Delta_2 : \mathrm{Tr}M_1M_2).$$

Note that $(Q \times Q)^{ss} \backslash (Q \times Q)^{nn}$ exactly corresponds to pairs of matrices (M_1, M_2) of which at least one is nilpotent, but $\mathrm{Tr}M_1M_2 \neq 0$. Under this map ϕ, these points are sent exactly to the point $(0 : 0 : 0 : 0 : 1)$ we excluded: hence,

$$\phi : X^{ss} \to \left\{(x_1 : x_2 : x_3 : x_4 : x_5) \in \mathbb{P}^4 \,|\, x_1x_2 = x_3x_4\right\}$$

is a well-defined map. However, it does not seem to be a quotient because it does not separate the various different orbits in X^{ss}.

We now also know that if we would know the structure of the ring of invariants of the affine part defined by $\mathrm{Tr}M_1M_2 \neq 0$, we would have covered the complete $(Q\times Q)^{ss}$. However, we also note that finding the structure of this ring of invariants might not be so simple: using all the obvious invariants we do not seem to have succeeded in building a quotient for $(Q\times Q)^{ss}$. Also, building such a quotient seems to boil down to "blowing up" the map ϕ at the image point $(0:0:0:0:1)$, of which it does not seem obvious it this can be done.

Now we look at how $(Q\times Q)^{ss}$ behaves under ϕ. In [**2**, p.18] it is proved that a matrix pair (M_1, M_2) being irreducible is equivalent to $\mathrm{Tr}((M_1M_2)^2 - M_1^2M_2^2) \neq 0$. One checks that this can be written in terms of elements of the algebra of invariants as follows:

$$\begin{aligned}\mathrm{Tr}((M_1M_2)^2 - M_1^2M_2^2) \;=\;& (\mathrm{Tr}M_1)^2\Delta_2^2 + \Delta_1^2(\mathrm{Tr}M_2)^2 + (\mathrm{Tr}M_1M_2)^2\\ & -\mathrm{Tr}M_1\mathrm{Tr}M_2\mathrm{Tr}M_1M_2 - 4\Delta_1^2\Delta_2^2;\end{aligned}$$

see [**9**] for how we obtained this relation. We see that X^s maps to the open subset

$$Y' = \{(x_1:x_2:x_3:x_4:x_5) \in \mathbb{P}^5 \mid x_1^2 + x_2^2 + x_5^2 - x_3x_5 - 4x_4^2 \neq 0\} \subset Y.$$

But then by Proposition 25, we get the following result:

PROPOSITION 33. *Let Y' be the variety:*

$$\{(x_1:x_2:x_3:x_4:x_5) \mid x_1x_2 = x_3x_4,\ x_1^2 + x_2^2 + x_5^2 - x_3x_5 - 4x_4^2 \neq 0\} \subset \mathbb{P}^5$$

The map ϕ defined by (3.5.1) is a geometric quotient $X^s \to Y'$.

3.6. Stability for more than 2 matrices

We now find a criterium for (semi-)stability in a k-fold product of Q's, with $k > 2$. In this case, we get an embedding

$$\begin{aligned} P &= Q\times \ldots \times Q\\ &= \oplus_{i=1,..k}\left\{(a_i:b_i:c_i:d_i:\Delta_i) \mid a_id_i - b_ic_i = \Delta_i^2\right\}\\ &\subset \mathbb{P}^4\times\ldots\times\mathbb{P}^4 \hookrightarrow \mathbb{P}^{5^k-1}\end{aligned}$$

with basis consisting of k-fold products of $a_i, b_i, c_i, d_i, \Delta_i$'s for $i = 1, ..., k$. We can proceed in the same way of Lemma 30 to find the unstable points of P. The criterion we get is slightly awkward:

PROPOSITION 34. *A point $x = (M_1, ..., M_k)$ is unstable if and only if for some i, matrix M_i is nilpotent and when x is conjugated so that M_i is in Jordan form, the number of matrices with bottom-left element non-zero is smaller than the number of nilpotent matrices commuting with M_i.*

PROOF. A point x is unstable if there is a one-parameter subgroup λ, which we can assume to acts as $\mathrm{diag}(\lambda, \lambda^{-1})$, so that all weights with respect to λ are strictly positive. Then for any combination of a, c, d, Δ,

$$\prod_{j=1,..,k}\{a_j, c_j, d_j, \Delta_j\} = 0.$$

But this means that at least one matrix M_i is nilpotent: for if we could choose an $a_i, c_i, d_i, \Delta_i \neq 0$ for any i, then its product would be non-zero as well. So we can conjugate x so that M_i is in Jordan form.

But then consider the $(k-1)$-tuple $\hat{x}$ we get if we remove M_i from x: then x is unstable exactly when all weights of $\hat{x}$ with respect to λ must be non-negative. This is automatically satisfied if all $c_j = 0$. On the other hand, if $c_j \neq 0$ for some

j, then consider the $(k-2)$-tuple $\tilde{x}$ we get if we remove M_i, M_j from x. Then x is unstable exactly when all weights of $\tilde{x}$ with respect to λ are strictly positive.

But this is the case we started with, so inductively we obtain the statement we want to prove. □

PROPOSITION 35. *A point $x = (M_1, ..., M_k)$ is semi-stable if for any of its nilpotent matrices M_i, when we conjugate x so that M_i is in Jordan form, the number of matrices with bottom-left element non-zero is at least the number of nilpotent matrices commuting with M_i.*

We notice that the reasoning we used to find the stable points in the two-matrix setting also works for an arbitrary number of matrices:

PROPOSITION 36. *A point $x = (M_1, ..., M_k)$ is stable if and only if it is irreducible and none of the matrices is nilpotent.*

PROOF. Such a tuple is not stable if and only if there is one-parameter subgroup λ so that all weights with respect to λ are ≥ 0. First suppose we have a matrix tuple $(M_1, ..., M_k)$ that is not stable. Using the above notation of $M_i = (a_i : b_i : c_i : d_i : \Delta_i)$, we get that

$$c_i \cdot \prod_{j \neq i} \{a_j, c_j, d_j, \Delta_j\} = 0$$

So suppose we have some $c_i \neq 0$. Then for one j we must have $a_j = c_j = d_j = \Delta_j = 0$; for suppose for all j we that one these four entries is non-zero, then the product of all these choices would be non-zero as well. On the other hand, all $c_i = 0$ gives a reducible pair. Clearly the other direction of the proposition also holds. □

3.7. Constructing a quotient for $Q \times Q \times Q$

We will now look at the case with 3 matrices. Recall that in the affine version of this case, apart from the usual degree 1 and degree 2 generators for the algebra of invariants, we have one degree 3 generator, and one relation holding between the generators.

Similarly to the $Q \times Q$ case, we can look at our affine covering 2 and see what maps the invariant functions induce. For example, for the embedding of $SL_2 \times SL_2 \times SL_2$, we get the following induced map on $Q \times Q \times Q$:

$$\begin{aligned}((M_1 : \Delta_1), (M_2 : \Delta_2), (M_3 : \Delta_3)) \quad \mapsto \quad & (\mathrm{Tr} y_1 \Delta_2 \Delta_3 : \Delta_1 \mathrm{Tr} y_2 \Delta_3 : \Delta_1 \Delta_2 \mathrm{Tr} y_3 : \\ & \mathrm{Tr} y_1 y_2 \Delta_3 : \Delta_2 \mathrm{Tr} y_1 y_3 : \Delta_1 \mathrm{Tr} y_2 y_3 : \\ & \mathrm{Tr}(y_1 y_2 y_3 - y_1 y_3 y_2) : \Delta_1 \Delta_2 \Delta_3).\end{aligned}$$

Indeed, considering all embeddings, we get all the "obvious" polynomials linear in each of the three matrices: the eight combinations

$$i_1, ..., i_8 = \{\mathrm{Tr} M_1, \Delta_1\} \cdot \{\mathrm{Tr} M_2, \Delta_2\} \cdot \{\mathrm{Tr} M_3, \Delta_3\},$$

the combinations with traces of two matrices

$$i_9, i_{10} = \mathrm{Tr} M_1 M_2 \cdot \{\mathrm{Tr} M_3, \Delta_3\};\ i_{11}, i_{12} = \mathrm{Tr} M_1 M_3 \cdot \{\mathrm{Tr} M_2, \Delta_2\};$$
$$i_{13,14} = \mathrm{Tr} M_2 M_3 \cdot \{\mathrm{Tr} M_1, \Delta_1\},$$

and finally $i_{15} = (\mathrm{Tr} M_1 M_2 M_3 - M_1 M_3 M_2)$. In all, this gives us a function ϕ:$(Q \times Q \times Q)^{nn} \to \mathbb{P}^{14}$ which we would hope to be a good quotient on its image:

$$\begin{aligned}(M_1 : \Delta_1, M_2 : \Delta_2, M_3 : \Delta_3) \quad \overset{\phi}{\mapsto} \quad & (tr_1 tr_2 tr_3 : tr_1 tr_2 \Delta_3 : tr_1 \Delta_2 tr_3 : tr_1 \Delta_2 \Delta_3 : \\ & \Delta_1 tr_2 tr_3 : \Delta_1 tr_2 \Delta_3 : \Delta_1 \Delta_2 tr_3 : \Delta_1 \Delta_2 \Delta_3 : \\ & tr_{12} tr_3 : tr_{12} \Delta_3 : tr_{13} tr_2 : tr_{13} \Delta_2 : \\ & tr_{23} tr_1 : tr_{23} \Delta_1 : tr_{123}).\end{aligned}$$

For this, we need to check that when restricted to any affine part of the cover, the image is isomorphic to the good quotient induced by its algebra of invariants. But for example, on the affine part of $Q \times Q \times Q$ where $\Delta_1 = \Delta_2 = \Delta_3 = 1$,we see that ϕ reduces to the following map:

$$\begin{aligned}(M_1 : \Delta_1, M_2 : \Delta_2, M_3 : \Delta_3) \quad \mapsto \quad & (tr_1 tr_2 tr_3 : tr_1 tr_2 : tr_1 tr_3 : tr_1 : \\ & tr_2 tr_3 : tr_2 : tr_3 : 1 : \\ & tr_{12} tr_3 : tr_{12} : tr_{13} tr_2 : tr_{13} : \\ & tr_{23} tr_1 : tr_{23} : tr_{123}),\end{aligned}$$

so its image is clearly isomorphic to $\mathbb{C}[SL_2^3]$. This motivates the claim that:

CLAIM 37. The map $\phi : (Q \times Q \times Q)^{nn} \to \phi((Q \times Q \times Q)^{nn})$, as defined above, is a good quotient.

We did not check the restrictions of ϕ to the other affine parts, but one would not expect problems in this verification.

Similarly to the $Q \times Q$ case, we have the situation that

$$(Q \times Q \times Q)^{ss} \subsetneq (Q \times Q \times Q)^{nn} \subsetneq (Q \times Q \times Q)^{s},$$

and again, the map ϕ is exactly defined in $(Q \times Q \times Q)^{ss}$, as the following Lemma shows:

LEMMA 38. *A matrix triple* $p = (M_1 : \Delta_1, M_2 : \Delta_2, M_3 : \Delta_3) \in Q \times Q \times Q$ *is semi-stable if and only if the map* ϕ *above is well-defined in* p.

PROOF. First we note by direct calculation that all invariants are zero for the types of unstable triples described above. Conversely, suppose all invariant function are zero. We want to show that we have an unstable point.

Note that one of the three matrices has to be nilpotent because of the invariants $i_1, ..., i_8$: say it is matrix M_3. Notice that now directly all $i_1, ..., i_8$ are zero. We can now write M_3 in Jordan form so that $a_3 = c_3 = d_3 = \Delta_3 = 0$; $b_3 \neq 0$.

Also, one calculates that $\mathrm{Tr} M_1 M_3 = c_1 b_3$. Because $b_3 \neq 0$, if we look at i_{11} and i_{12}, then either M_2 is nilpotent or $c_1 = 0$. In the same way, either M_1 is nilpotent or $c_2 = 0$. If $c_1 = c_2 = 0$ then the situation is easy: the triple is reducible and we have an unstable point by our criterion.

Otherwise, suppose $c_1 \neq 0$, but M_2 is nilpotent and $c_2 = 0$. One calculates that

$$B \cdot C - C \cdot B = \begin{pmatrix} 0 & (d_2 - a_2) b_3 \\ 0 & 0 \end{pmatrix}; \ \mathrm{Tr} M_1 M_2 M_3 = c_1 (d_2 - a_2) b_3 = 0,$$

so in fact B and C commute and we have an unstable point by our criterion.

Finally, consider the case where $c_1, c_2 \neq 0$, and M_1 and M_2 are both nilpotent. We claim that M_1 and M_2 commute, ie,

$$C := M_1 \cdot M_2 - M_2 \cdot M_1 = \begin{pmatrix} -b_2 c_1 + b_1 c_2 & -2 a_2 b_1 + 2 a_1 b_2 \\ 2 a_2 c_1 - 2 a_1 c_2 & b_2 c_1 - b_1 c_2 \end{pmatrix} = 0.$$

The only invariant we can still use, i_{15}, gives us that $a_1c_2 = a_2c_1$, so $C_{21} = 0$. But nilpotency of M_1, M_2 gives $a_1^2 = b_1c_1$ and $a_2^2 = b_2c_2$, so

$$\begin{aligned}
C_{11} &= -b_2c_1 + b_1c_2 = -\frac{a_2^2c_1}{c_2} + \frac{a_1^2c_2}{c_1} = \frac{a_1^2c_2^2 - a_2^2c_1^2}{c_1c_2} \\
&= \frac{(a_1c_2 - a_2c_1)(a_1c_2 + a_2c_1)}{c_1c_2} = 0; \\
C_{12} &= -2a_2b_1 + 2a_1b_2 = -2\frac{a_1b_1c_2}{c_1} + 2\frac{a_1b_2c_1}{c_1} = -2\frac{a_1}{c_1}(-b_2c_1 + b_1c_2) = 0.
\end{aligned}$$

So in this case our criterion is also satisfied. This concludes our proof. □

There are still a few pices of the puzzle missing for the $m = 3$ case. First, we could calculate the image under ϕ of $(Q \times Q \times Q)^{ss}$ and $(Q \times Q \times Q)^{nn}$. The above lemma would suggest that the $M = \phi((Q \times Q \times Q)^{ss})$ is the complete space induced by the invariants with the obvious relations holding between them:

$$\begin{aligned}
M \;=\; \{&(i_1 : i_2 : \ldots : i_{15}) \in \mathbb{P}^{14} \,| \\
&i_1i_4 = i_2i_3, i_1i_6 = i_2i_5, i_1i_7 = i_3i_5, i_2i_8 = i_4i_6, i_3i_8 = i_4i_7, i_5i_8 = i_6i_7, \\
&i_1i_8 = i_2i_7 = i_3i_6 = i_4i_5, \\
&i_9i_1 = i_{10}i_2, i_9i_3 = i_{10}i_4, i_9i_5 = i_{10}i_6, i_9i_7 = i_{10}i_8, \\
&i_{11}i_1 = i_{12}i_3, i_{11}i_{11}i_2 = i_{12}i_4, i_{11}i_5 = i_{12}i_7, i_{11}i_6 = i_{12}i_8, \\
&i_{13}i_1 = i_{14}i_5, i_{13}i_2 = i_{14}i_6, i_{13}i_3 = i_{14}i_7, i_{13}i_4 = i_{14}i_8, \\
&i_8^2i_{15}^2 = i_4^2i_{12}^2 + \ldots + 4i_8i_{10}i_{14}\}.
\end{aligned}$$

Here, the last relation a projective version of our relation (1.3.3) that holds in the case of three affine matrices (page 22). For the image of $(Q \times Q \times Q)^{nn}$ the closed set

$$M' = \{(i_1 : i_2 : \ldots : i_{15}) \in \mathbb{P}^{14} \,|\, i_1 = i_2 = \ldots = i_8 = 0\}$$

would then need to be removed. We did not verify this.

Finally, one could check whether

$$\phi((Q \times Q \times Q)^s)$$

is open in

$$\phi((Q \times Q \times Q)^{nn})$$

to be able to conclude that on the stable points, ϕ is in fact a geometric quotient.

For more than three matrices, in general it seems doable to formulate how a good quotient on the non-nilpotent matrices should look like; however, calculating the image space and the image of the stable points would again be difficult. Also, it is not clear whether the proof of Lemma 38 easily generalizes to more matrices.

3.8. Orbits in the affine case

We now do some calculations on orbits of SL_2 matrices in the affine coordinate ring, and compare this to our above results.

First, we look at the 3 matrix-case. Corollary 18 tells us that $\mathbb{C}[SL_2 \times SL_2 \times SL_2]^{SL_2}$ is a quotient ring with 7 generators a, b, c, d, e, f, g and one relation, $r \in \mathbb{C}[a, b, c, d, e, f, g]$ – see equation (1.3.3). The *singular locus* of this quotient is the set of points p in the corresponding algebraic variety where

$$\mathrm{grad}_p r = \left[\frac{\partial r}{\partial a}(p), \ldots, \frac{\partial r}{\partial g}(p)\right] = 0;$$

these are thus the points in the variety where no tangent plane can be defined. We study the relation between points in the singular locus and unstable points in the corresponding projective embedding above.

One can calculate the singular locus of the quotient using the Macaulay software package[1]. Using our familiar shorthands $tr_i = TrM_i$, $tr_{ij} = TrM_iM_j$, $tr_{ijk} = Tr(M_iM_jM_k - M_iM_kM_j)$, we get that the singular locus is the following ideal (I):

$$\begin{aligned}
(I) \quad = \quad & (\mathrm{tr}_{123}, \\
& -\mathrm{tr}_1\mathrm{tr}_2\mathrm{tr}_{12} + \mathrm{tr}_1^2 + \mathrm{tr}_2^2 + \mathrm{tr}_{12}^2 - 4, \\
& -\mathrm{tr}_2\mathrm{tr}_3\mathrm{tr}_{23} + \mathrm{tr}_2^2 + \mathrm{tr}_3^2 + \mathrm{tr}_{23}^2 - 4, \\
& -\mathrm{tr}_1\mathrm{tr}_3\mathrm{tr}_{13} + \mathrm{tr}_1^2 + \mathrm{tr}_3^2 + \mathrm{tr}_{13}^2 - 4, \\
& -\mathrm{tr}_1^2\mathrm{tr}_3 + \mathrm{tr}_2^2\mathrm{tr}_3 - \mathrm{tr}_2\mathrm{tr}_{12}\mathrm{tr}_{13} + \mathrm{tr}_1\mathrm{tr}_{12}\mathrm{tr}_{23} + 2\mathrm{tr}_1\mathrm{tr}_{13} - 2\mathrm{tr}_2\mathrm{tr}_{23}, \\
& -\mathrm{tr}_1\mathrm{tr}_2^2 + \mathrm{tr}_1\mathrm{tr}_3^2 - \mathrm{tr}_3\mathrm{tr}_{12}\mathrm{tr}_{23} + \mathrm{tr}_2\mathrm{tr}_{13}\mathrm{tr}_{23} + 2\mathrm{tr}_2\mathrm{tr}_{12} - 2\mathrm{tr}_3\mathrm{tr}_{13}, \\
& -\mathrm{tr}_1^2\mathrm{tr}_2 + \mathrm{tr}_2\mathrm{tr}_3^2 - \mathrm{tr}_3\mathrm{tr}_{12}\mathrm{tr}_{13} + \mathrm{tr}_1\mathrm{tr}_{13}\mathrm{tr}_{23} + 2\mathrm{tr}_1\mathrm{tr}_{12} - 2\mathrm{tr}_3\mathrm{tr}_{23}, \\
& \mathrm{tr}_1\mathrm{tr}_2^2\mathrm{tr}_3 - \mathrm{tr}_2\mathrm{tr}_3\mathrm{tr}_{12} - \mathrm{tr}_2^2\mathrm{tr}_{13} - \mathrm{tr}_1\mathrm{tr}_2\mathrm{tr}_{23} - 2\mathrm{tr}_1\mathrm{tr}_3 + 2\mathrm{tr}_{12}\mathrm{tr}_{23} + 4\mathrm{tr}_{13}, \\
& \mathrm{tr}_1^2\mathrm{tr}_2\mathrm{tr}_3 - \mathrm{tr}_1\mathrm{tr}_3\mathrm{tr}_{12} - \mathrm{tr}_1\mathrm{tr}_2\mathrm{tr}_{13} - \mathrm{tr}_1^2\mathrm{tr}_{23} - 2\mathrm{tr}_2\mathrm{tr}_3 + 2\mathrm{tr}_{12}\mathrm{tr}_{13} + 4\mathrm{tr}_{23}, \\
(3.8.1) \quad & \mathrm{tr}_1\mathrm{tr}_2\mathrm{tr}_3^2 - \mathrm{tr}_2\mathrm{tr}_3\mathrm{tr}_{13} - \mathrm{tr}_3^2\mathrm{tr}_{12} - \mathrm{tr}_1\mathrm{tr}_3\mathrm{tr}_{23} - 2\mathrm{tr}_1\mathrm{tr}_2 + 2\mathrm{tr}_{13}\mathrm{tr}_{23} + 4\mathrm{tr}_{12}).
\end{aligned}$$

We can interpret at least the equations 2-4 of this ideal by noting that for matrices in SL_2, we have

$$\mathrm{Tr}\left((M_1M_2)^2 - M_1^2M_2^2\right) = -\mathrm{tr}_1\mathrm{tr}_2\mathrm{tr}_{12} + \mathrm{tr}_1^2 + \mathrm{tr}_2^2 + \mathrm{tr}_{12}^2 - 4.$$

Other than that, what seems to happen is that in the coordinate ring of the singular locus, we get $\mathbb{C}[\mathrm{tr}_1, \mathrm{tr}_2, \mathrm{tr}_3]$ and then we get degree 2 extensions for the tr_{ij} from relations (2-4), with relations (5-10) adding some conditions that limit the degree of the overall extension.

For example, if one looks at the coordinate ring as an extension of

$$\mathbb{C}[\mathrm{tr}_1, \mathrm{tr}_2, \mathrm{tr}_3, \mathrm{tr}_{12}],$$

then at first glance equations (8-9) look like they can be solved uniquely for tr_{13} and tr_{23}; however, careful inspection gives that in fact the discriminant of the system of equations is a polynomial in (I), so this does not happen. We did not look into this matter further.

We have the following criterion for points to lie in the singular locus:

LEMMA 39. *A matrix triple $(A, B, C) \in SL_2^{\times 3}$ is in this singular locus (I) defined by (3.8.1) if and only if it is reducible.*

PROOF. We note that reducible matrix triples are in the singular locus. Conversely, take a triple of SL_2 matrices

$$\left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, C = \begin{pmatrix} i & j \\ k & l \end{pmatrix}\right)$$

in the singular locus.

First suppose one matrix, say A, can be diagonalized to $A = \mathrm{diag}(\lambda, \lambda^{-1})$. Looking at the second and fourth relations of (I), one finds that either $\lambda = \pm 1$, or $eh = il = 1$.

If $\lambda = \pm 1$, then the third relation is the only remaining one. Suppose we can diagonalize B, then this relation gives either $B = \pm \mathrm{Id}$ (so directly we have a reducible triple), or $il = 1$. But then because $il - jk = 1$, either $k = 0$ or $j = 0$ and we can swap the eigenvalues of B to see that the triple is reducible.

Otherwise, we have $eh = il = 1$, and so $fg = jk = 0$. If $gk = 0$ then we are done and if $fj = 0$ then we can swap the eigenvalues of A to see that the triple is

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[1] See `http://www.math.uiuc.edu/Macaulay2/`; we used the `singularLocus` command to calculate the locus, and `decompose` to write it in a more recognizable form.

reducible. If $gj = 0$, then the first relation gives $fk = 0$. This means that one of the B, C is diagonalizable: a case we already handled. The case $fk = 0$ is similar. This finishes the case when we have a diagonalizable matrix.

So now suppose neither matrix is diagonalizable. Writing A in Jordan form we get $a = b = d = 1, c = 0$. Relations 2 and 4 now give $g = k = 0$, so we are done. ☐

REMARK 40. • By Corollary 36, a matrix triple $(A, B, C) \in SL_2^{\times 3}$ is not stable if and only if it is reducible. Thus, the matrices in the singular locus correspond exactly to points in the embedding that are not stable.

• In [**2**, p.18-20], it is shown for general 2×2 matrices that (A, B) is reducible if and only if $\mathrm{Tr}(AB)^2 = \mathrm{Tr}A^2B^2$, and that (A, B, C) is reducible if and only if (A, B), (B, C) and (A, C) are reducible and $\mathrm{Tr}ABC = \mathrm{Tr}ACB$. This corresponds exactly to our above result.

We have now found a criterium for a matrix triple to be in the singular locus for the $SL_2^{\times 3}$ case: this corresponds to the point not being stable. Now, recall that a point being stable by definition means that its orbit is closed and the dimension of the orbit is maximal. The following two properties also holds between the various properties:

LEMMA 41. [**2**, p.25] *Let $x = (M_1, ..., M_k) \in M_{2\times 2}^k$. Then the orbit of x is closed if and only if either $(M_1, ..., M_k)$ is irreducible or $(M_1, ..., M_k)$ is simultaneously diagonalizable.*

PROOF. See [**2**, p.25]. ☐

LEMMA 42. *Let $x = (M_1, M_2, ..., M_n) \in M_{2\times 2}^k$. Then the orbit of x under simultaneous conjugation is smaller than 3 if and only if all matrices commute.*

PROOF. Suppose we know the that dimension of the orbit of one matrix, say M_1, under conjugation equals d. Then the orbit of (M_1, M_2) under conjugation is also d if and only if for any similarity matrix Q, the implication

$$QM_1Q^{-1} = M_1 \Rightarrow QM_2Q^{-1} = M_2 \tag{3.8.2}$$

holds. This is because should the dimension be larger than d, we need to have $\hat{Q}M_1\hat{Q}^{-1} = \tilde{Q}M_1\tilde{Q}^{-1}$ but $\hat{Q}M_2\hat{Q}^{-1} \neq \tilde{Q}M_2\tilde{Q}^{-1}$ for some $\hat{Q}$, $\tilde{Q}$. But conjugating with $\tilde{Q}^{-1}$ then gives $\tilde{Q}^{-1}\hat{Q}M_1\hat{Q}^{-1}\tilde{Q} = M_1$ but $\tilde{Q}^{-1}\hat{Q}M_2\hat{Q}^{-1}\tilde{Q} \neq M_2$, so (3.8.2) is broken for $Q = \tilde{Q}^{-1}\hat{Q}$.

Now, all matrices commuting means that either all matrices are simultaneously diagonalizable (if one matrix is diagonalizable, then for the other matrices to commute with it, they must be diagonalizable as well), or they can all be simultaneously conjugated to upper triangular form with ± 1 on the diagonals (by the same argument).

In the first case, diagonalize the matrices; then they have common stabilizer $S_1 = \{\mathrm{diag}(\lambda, \lambda^{-1}) \mid \lambda \neq 0\}$; in the second case, the matrices have common stabilizer

$$S_2 = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathbb{C} \right\}; \tag{3.8.3}$$

in both cases the condition (3.8.2) clearly holds. Since the orbit of any one matrix is at most 2-dimensional (it is limited by the invariance of the trace and determinant), the orbit of the tuple must be at most 2-dimensional as well.

Conversely, suppose the orbit of some tuple x has dimension smaller than 3. Suppose one matrix, say M_1, can be diagonalized to $\mathrm{diag}(\lambda, \lambda^{-1})$ for some $\lambda \neq \pm 1$.

Then already the orbit of x is 2. For the dimension of the orbit of x to be ≤ 2, then all other matrices must be stabilized by S_1. But as we know these matrices act as

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \lambda^2 b \\ \lambda^{-2} c & d \end{pmatrix},$$

so we must have $b = d = 0$, and all other matrices are diagonal matrices and commute withM_1.

Now suppose one matrix M_1 cannot be diagonalized, so its Jordan form is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The orbit of this matrix is also 2: it consist of all matrices with trace 2 and determinant 1 other than the identity. Then for the dimension of the orbit of x to be ≤ 2, all other matrices must be stabilized by S_2. One checks that this means that all matrices must be upper triangular matrices with ± 1 on the diagonal, commuting with M_1.

All this leaves is the trivial case of all matrices being equal to $\pm\mathrm{Id}$. This concludes the proof. $\square$

Combining everything, we get:

PROPOSITION 43. *For any k-tuple of matrices in SL_2, we have the following set of equivalences and consequences:*

$$\begin{aligned} \neg\textit{stable} \Leftrightarrow \textit{reducible} &\Leftrightarrow \textit{commute} \wedge \neg\textit{closed} \\ \textit{closed} &\Leftrightarrow \neg\textit{reducible} \vee \textit{simult. diag} \\ \textit{simult. diag} &\Rightarrow \textit{commute.} \end{aligned}$$

For a 3-tuple, we know that reducibility is equivalent to the tuple being mapped to the singular locus of the algebra of invariants.

CHAPTER 4

The matrix space as a representation

We now look at our space of matrices with the SL_2 action in the general context of SL_2 representations. First, we very briefly describe how the representation theory of SL_2, and, indeed, representation theory itself, works. Then we describe what happens in our situation.

4.1. Representation theory of SL_2

4.1.1. Representation theory: a crash course. One main goal of representation theory is to classify, up to isomorphism, all possible representations of a given group. We introduce some basic notions.

If W is a subspace of V that is closed under G, then W is also a G-representation: it is called a *subrepresentation* of G. A G-module V has at least two subrepresentations: the *trivial representation* $\{0\}$ and V itself. If these are all subrepresentations of V, then V is said to be *irreducible*.

A G-representation V is reducible if we can write it as a direct sum $V = W_1 \oplus W_2$ of G-representations (of course, $W_1 \cong W_1 \oplus \{0\}$ and $W_2 \cong \{0\} + W_2$ are then subrepresentations of G, so V is not irreducible). A G-representation V is called *completely reducible* if it can be written as a direct sum of irreducible G-representations.

If G satisfies certain conditions, then in this way we can write any finite-dimensional representation of G as a direct sum of irreducible representations. This way of writing is then also unique up to isomorphism. Representation theory, then, is concerned with determining, up to isomorphism, all these irreducible representations of a given group.

4.1.2. Irreducible representations of SL_2. The representation theory of SL_2 is well-known and quite simple. Any finite-dimensional SL_2-representation can be written as a unique direct sum of irreducible representations. Also, for any $0 \leq n \in \mathbb{N}$ there is a unique irreducible representation of dimension $n+1$, which is denoted $[n]$. This representation can be seen as

$$[n] = \{f \in \mathbb{C}[x_1, x_2] \,|\, f \text{ is homogeneous of degree } n\};$$

SL_2 acts on this in the obvious way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x_1 = ax_1 + cx_2; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x_2 = bx_1 + dx_2.$$

When writing down direct sums of these irreducible representations, often the direct sum symbol $\oplus$ is denoted by $+$, and a direct sum of k times the irreducuble representation $[l]$ is often denoted as $k \cdot [l]$. For instance, we could write $[0] \oplus [0] \oplus [4] \oplus [6]$ as $2 \cdot [0] + [4] + [6]$.

4.1.3. Some isomorphic representations. We now look at the action of SL_2 on vector spaces of matrices by simultaneous conjugation. We denote this action by $\diamond$, eg:

$$G \diamond \begin{pmatrix} a & b \\ c & d \end{pmatrix} = G \begin{pmatrix} a & b \\ c & d \end{pmatrix} G^{-1}.$$

PROPOSITION 44. *As a SL_2 representation by simultaneous conjugation M_0 is isomorphic to* $[2]$.

PROOF. One checks that the following linear map $\phi : M_0 \to [2]$ gives an isomorphism:

$$\begin{aligned} M_2^0 &\cong [2] \\ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} &\leftrightarrow x_1^2 \\ \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &\leftrightarrow x_1x_2 \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &\leftrightarrow x_2^2. \end{aligned}$$

For instance, one checks that

$$\begin{aligned} \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \diamond \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\right) &= \phi\begin{pmatrix} ac & -a^2 \\ c^2 & ac \end{pmatrix} = a^2 \cdot x_1^2 + 2ac \cdot x_1x_2 + c^2 \cdot \phi x_2^2 \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x_1^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \phi \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

□

PROPOSITION 45. *The following isomorphism of SL_2-representations holds:*

$$[2] \otimes [2] \cong [0] + [2] + [4].$$

PROOF. We introduce the following notation for the basis elements:

$$[2] \otimes [2] = \operatorname{span}\{x_i \otimes x_j\};\ [2] = \operatorname{span}\{y_1^2, y_1y_2, y_2^2\};\ [4] = \operatorname{span}\{z_1^k z_2^{4-k}\}.$$

We can then check that the following linear map $\phi : [0] + [2] + [4] \to [2] \otimes [2]$ defines an isomorphism of SL_2-representations:

$$\begin{aligned} [0] + [2] + [4] &\cong [2] \otimes [2] \\ 1 &\leftrightarrow x_1^2 \otimes x_2^2 + x_2^2 \otimes x_1^2 - 2x_1x_2 \otimes x_1x_2 \\ y_1^2 &\leftrightarrow x_1^2 \otimes x_1x_2 - x_1x_2 \otimes x_1^2 \\ y_1y_2 &\leftrightarrow \frac{1}{2}(x_1^2 \otimes x_2^2 - x_2^2 \otimes x_1^2) \\ y_2^2 &\leftrightarrow x_1x_2 \otimes x_2^2 - x_2^2 \otimes x_1x_2 \\ z_1^4 &\leftrightarrow x_1^2 \otimes x_1^2 \\ z_1^3z_2 &\leftrightarrow \frac{1}{2}(x_1^2 \otimes x_1x_2 + x_1x_2 \otimes x_1^2) \\ z_1^2z_2^2 &\leftrightarrow \frac{1}{6}(x_1^2 \otimes x_2^2 + 4 \cdot x_1x_2 \otimes x_1x_2 + x_2^2 \otimes x_1^2) \\ z_1z_2^3 &\leftrightarrow \frac{1}{2}(x_1x_2 \otimes x_2^2 + x_2^2 \otimes x_1x_2) \\ z_2^4 &\leftrightarrow x_2^2 \otimes x_2^2. \end{aligned}$$

For instance, for a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ one checks that

$$\begin{aligned}
\phi\left(M \cdot y_1^2\right) &= \phi(a^2 y_1^2 + 2ac y_1 y_2 + c^2 y_2^2) \\
&= a^2\left(x_1^2 \otimes x_1 x_2 - x_1 x_2 \otimes x_1^2\right) + 2ac \cdot \frac{1}{2}(x_1^2 \otimes x_2^2 - x_2^2 \otimes x_1^2) + \\
&\quad c^2\left(x_1 x_2 \otimes x_2^2 - x_2^2 \otimes x_1 x_2\right) \\
&= (a^2 bc - a^3 d) \cdot x_1 x_2 \otimes x_1^2 + (abc^2 - a^2 cd) x_2^2 \otimes x_1^2 + \\
&\quad (a^3 d - a^2 bc) x_1^2 \otimes x_1 x_2 + (bc^3 - ac^2 d) x_2^2 \otimes x_1 x_2 + \\
&\quad (a^2 cd - abc^2) x_1^2 \otimes x_2^2 + (ac^2 d - bc^3) x_1 x_2 \otimes x_2^2 \\
&= M \cdot (x_1^2 \otimes x_1 x_2 - x_1 x_2 \otimes x_1^2) \\
&= M \cdot \phi(y_1^2).
\end{aligned}$$

The calculations with the $z_1^k z_2^{4-k}$ basis elements are longer but similar. □

4.2. Representation theory of Q

We are styding

$$Q = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \Delta \,|\, ad - bc = \Delta^2 \right\} \subset \mathbb{P}^4.$$

By definition, to find its stable and semi-stable points, we need to look at the SL_2 action on the corresponding affine space in $V = \mathbb{A}^5$. We now do this by looking at this space as a SL_2 representaton. But we can write $V = M_2^0 \oplus \mathbb{C} \oplus \mathbb{C}$, and SL_2 acts trivially on the $\mathbb{C}$'s, so by Proposition 44 we have the isomorphism of SL_2 representations

$$V \cong [2] \oplus [0] \oplus [0].$$

Now again by Lemma 27, a point of V is unstable if and only if there is a one-parameter subgroup $\mathrm{diag}(t, t^{-1})$ with all positive weights.

But now note that the action of $\mathrm{diag}(t, t^{-1})$ on the irreducible SL_2-representations is easy to describe: we have

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} x_1^k x_2^l = t^{k-l} x_1^k x_2^l.$$

But then the only basis element of $[2] \oplus [0] \oplus [0]$ with positive weight is x_1^2, so the set of unstable points is $\{\lambda x_1^2 \,|\, \lambda \neq 0\}$; under our representation isomorphism this corresponds to the matrices

$$\left\{ \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} : 0 \,|\, \lambda \neq 0 \right\}.$$

This corresponds to our earlier results.

4.3. The $Q \times Q$-case

Similarly, $Q \times Q$ is a subset of $\mathbb{P}(V)$, with corresponding affine space V.

$$V = (M_2^0 \oplus \mathbb{C} \oplus \mathbb{C}) \otimes (M_2^0 \oplus \mathbb{C} \oplus \mathbb{C}).$$

4.3.1. An isomorphism of representations.

PROPOSITION 46. We have the following isomorphism of SL_2-representations:

$$V \cong [4] + 5 \cdot [2] + 5 \cdot [0].$$

Proof. We have:

$$\begin{aligned} V &\cong ([2] + 2 \cdot [0]) \otimes ([2] + 2 \cdot [0]) \\ &\cong [2] \otimes [2] + 2 \cdot [0] \otimes [2] + 2 \cdot [2] \otimes [0] + 4 \cdot [0] \otimes [0] \\ &\cong ([4] + [2] + [0]) + 2 \cdot [2] + 2 \cdot [2] + 4 \cdot [0] \\ &\cong [4] + 5 \cdot [2] + 5 \cdot [0]; \end{aligned}$$

here, we used Proposition 45 and the trivial representation isomorphisms $[0] \otimes V \cong V \cong V \otimes [0]$.

Let us write out this isomorphism ϕ. Let us write the basis elements of $[4] + 5 \cdot [2] + 5 \cdot [0]$ as $x_1^k x_2^{4-k}$, $y_i^k z_i^{2-k}$ $(1 \leq i \leq 4)$, v_i $(1 \leq i \leq 5)$. Using our earlier calculations, we can find the elements of V to which these basis elements correspond. Let us denote the basis elements of $M_2^0 \oplus \mathbb{C} \oplus \mathbb{C}$ as

$$\left\{ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \Delta \right\}.$$

Then the basis $x_1^k x_2^{4-k}$ corresponds to the following matrices: $\square$

$$\begin{aligned} x_1^4 &\leftrightarrow \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\ x_1^3 x_2 &\leftrightarrow \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\ x_1^2 x_2^2 &\leftrightarrow \frac{1}{6} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \frac{4}{6} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\ x_1 x_2^3 &\leftrightarrow \frac{1}{2} \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \right) \\ x_2^4 &\leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

There is one [2]-module coming from $[2] \otimes [2]$:

$$\begin{aligned} y_1^2 &\leftrightarrow \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\ y_1 z_1 &\leftrightarrow \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}) \\ z_1^2 &\leftrightarrow \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \end{aligned}$$

and four coming from $[2] \otimes [0]$ and $[0] \otimes [2]$. The basis elements $y_2^k z_2^{2-k}$ and $y_3^k z_3^{2-k}$ correspond to

$$\begin{aligned} y_{\{2,3\}}^2 &\leftrightarrow \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \otimes \left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \Delta \right\} \\ y_{\{2,3\}} z_{\{2,3\}} &\leftrightarrow \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \Delta \right\} \\ z_{\{2,3\}}^2 &\leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \Delta \right\}; \end{aligned}$$

the other two are symmetric. Finally, we have 5 [0]-modules. Four of them are from the traces and determinants of the matrices:

$$v_{\{1,2,3,4\}} = \left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \Delta \right\} \otimes \left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \Delta \right\},$$

and the last one comes from the $[2] \otimes [2]$:

$$v_5 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - 2 \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

For example, one can check that this element of V is indeed invariant: let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then:

$$\begin{aligned}
M \diamond v_5 &= M \diamond \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) + M \diamond \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) - 2M \diamond \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} ac & -a^2 \\ c^2 & -ac \end{pmatrix} \otimes \begin{pmatrix} bd & -b^2 \\ d^2 & -bd \end{pmatrix} + \begin{pmatrix} bd & -b^2 \\ d^2 & -bd \end{pmatrix} \otimes \begin{pmatrix} ac & -a^2 \\ c^2 & -ac \end{pmatrix} \\
&\quad -2 \begin{pmatrix} \frac{1}{2}(ad+bc) & -ab \\ cd & -\frac{1}{2}(ad+bc) \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2}(ad+bc) & -ab \\ cd & -\frac{1}{2}(ad+bc) \end{pmatrix} \\
&= (a^2b^2 + a^2b^2 - 2a^2b^2) \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) + \\
&\quad (a^2bd + b^2ac - ab(ad+bc)) \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \\
&\quad (a^2d^2 + b^2c^2 - 2abcd) \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) + \dots \\
&= 0 \cdot \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) + 0 \cdot \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + 1 \cdot \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) + \dots \\
&= v_5.
\end{aligned}$$

4.3.2. Unstable points. Now similarly to what we did above, we can find the set of unstable points of V: this is simply the union of all the unstable points in the irreducible subrepresentations. For the copy of [4] in V, this is $\mathrm{span}\{x_1^4, x_1^3x_2\}$, and for each copy of [2], we get $\mathrm{span}\{y_i^2\}$. In all, we get a 7-dimensional subspace of V of unstable points. We note that

$$\begin{aligned}
\phi(x_1^3x_2) &= \frac{1}{2}\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\right), \\
\phi(y_1^2) &= \frac{1}{2}\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\right).
\end{aligned}$$

But then we can take the sum and difference of these basis elements to obtain:

PROPOSITION 47. *Up to conjugation, the unstable points of V are:*

$$\begin{aligned}
V^u &= \mathit{span}\left\{\phi(x_1^4), \phi(x_1^3x_2) + \phi(y_1^2), \phi(x_1^3x_2) - \phi(y_1^2), \phi(y_2^2), \phi(y_3^2), \phi(y_4^2), \phi(y_5^2)\right\} \\
&= \mathit{span}\{\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \Delta, \\
&\qquad \Delta \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right)\}.
\end{aligned}$$

But note that this corresponds exactly to our results in Lemma 30.

4.3.3. Generalization to more matrices. When repeating the above calculations for more copies of Q, we need to repeat the argument of 46 to write the vector space in terms of irreducible representations. For example, for three Q's the calculation would look like:

$$\begin{aligned}
V &\cong ([2] + 2 \cdot [0]) \otimes ([2] + 2 \cdot [0]) \otimes ([2] + 2 \cdot [0]) \\
&\cong ([4] + 5 \cdot [2] + 5 \cdot [0]) \otimes ([2] + 2 \cdot [0]) \\
&\cong [4] \otimes [2] + 5 \cdot [2] \otimes [2] + 2 \cdot [4] + 10 \cdot [2] + 10 \cdot [0].
\end{aligned}$$

One see that this involves calculating more representation isomorphism as in 45. We did not look into this further.

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