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Invariants under simultaneous conjugation of SL_2 matrices

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$$\begin{tikzcd}
(Q \times \ldots \times Q)^{ss} \arrow[r, "\phi"] \arrow[dr, "\psi"'] & Y \arrow[d, "\theta"] \\
 & Z
\end{tikzcd}$$

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ABSTRACT. Given tuples of SL_2-matrices, one can look at which functions in their coordinate ring do not change when we simultaneously conjugate the matrices: these are called the *invariant functions*. Our interest in them is motivated by the fact that these tuples, up to simultaneous conjugation, occur as the so-called “monodromy group” of certain linear differential equations.

We will look at these invariant functions from three different perspectives. First, we employ classical invariant theory to find the structure of the space generated by these invariant functions. Next, we use geometric invariant theory to define a “quotient” using these invariant functions. Finally, we place the results in the more general setting of representation theory by looking at the structure of the space of matrices as a SL_2-representation.

Contents

Introduction

0.1. Motivation

In this section we give a motivation for the subject of this thesis from the theory of differential equations. Understanding this motivation is not necessary to read the rest of the thesis, so the reader not interested in differential equations may skip directly to section 0.2. The definitions in this section are mainly from [**5**, p.113-116] and [**15**].

Suppose we are given an order 2 linear ordinary differential equation (ODE):

$$\frac{d^2y}{dt^2} + A_1(t) \cdot \frac{dy}{dt} + A_2(t) \cdot y = 0,$$

where $A_i(t)$ are fractions in t. We can consider t to be a point in the Riemann sphere $T = \mathbb{C} \cup \{\infty\}$ by substituting $u = t^{-1}$ in the case $t = \infty$. Let S be the set of singularities of the ODE. This is the set of points where the A_i have poles, or where the A_i of the re-written ODE for $t = \infty$ have poles.

Let $S = \{t \in T \mid A_i(t) = 0 \ \exists i\}$. Then one can solve this ODE for y as an analytic function of t in a small neighborhood of any $t_0 \in T$ exactly when $t_0 \in T\backslash S$: in this case, we have a two-dimensional space of linearly independent solutions of the ODE, and we can form a basis y_1, y_2 and consider the matrix

$$M = \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}.$$

Now we can consider the so-called “fundamental group” of $T\backslash S$ in the point t_0, denoted $\pi_1(T\backslash S, t_0)$. This is the group of closed loops starting at t_0 (functions $\lambda : [0,1] \to T\backslash S$ such that $\lambda(0) = \lambda(1) = t_0$) modulo homotopy. If we have n singular points, we get a free group with n generators g_i. However, if we run around all singularities once, then due to the spherical structure of the Riemann sphere, this is equivalent to running around no singularities at all, so we see that

$$\pi_1(T\backslash S, t_0) = (\text{free group on } g_1, ..., g_n)/(g_1 g_2 ... g_n = 1), \tag{0.1.1}$$

or, equivalently, it is the free group on $n-1$ generators.

Now let a point $t_0 \in T\backslash S$ and basis M of the solution space of the ODE be given. Given a loop λ, we can analytically continue any solution y to the ODE along this loop, giving us a basis of the solution space at any point on the loop. Denote this basis by $M(t_0, M, \lambda, t)$. It is known that if λ is a trivial loop (i.e., one that does not run around some $s \in S$), then $M(t_0, M, \lambda, 0) = M(t_0, M, \lambda, 1)$.

However, if we run around some singularity, then generally this is not the case: $M(t_0, M, \lambda, 1)$ gives a different basis of the solution space at t_0, and we can write this transformation of basis as $M(t_0, M, \lambda, 1) = Q \cdot M(t_0, M, \lambda, 0)$. Since this Q we get only depends on the homotopy class of the loop λ, this defines a map $\pi_1(T\backslash S, t_0) \to GL_2(\mathbb{C})$. Under suitable conditions for the ODE, one can show that in fact the $\det Q = 1$, so it is a map $\pi_1(T\backslash S, t_0) \to SL_2(\mathbb{C})$.

But then in light of (0.1.1), we can represent $\pi_1(T\backslash S, t_0)$ as an element of

$$\{(A_1, A_2, ..., A_n) \in SL_2^n \mid A_1 A_2 ... A_n = 1\},$$

or equivalently, as an element of $\{(A_1, A_2, ..., A_{n_1}) \in SL_2^{n-1}\}$. This result still depends on the choice of the initial basis of M, so for any change of basis Q, we want to have the equivalence under simultaneous conjugation

$$(A_1, ..., A_{n-1}) \sim (QA_1Q^{-1}, ..., QA_{n-1}Q^{-1}).$$

Thus, we are motivated to look at the equivalence classes (the so-called *orbits*) of tuples under the relation $\sim$ above:

$$\{(A_1, A_2, ..., A_{n-1}) \in SL_2^{n-1}\}/\sim . \tag{0.1.2}$$

Seeing simultaneous conjugation as an action of SL_2 on SL_2^{n-1}, this is the so-called *orbit space* of SL_2^{n-1} under the action of SL_2. Interpreting this orbit space is the central topic of this thesis.

0.2. Overview

The goal of this thesis is to study the object (0.1.2). We will do this in several different ways.

First, we will use invariant theory. This means we look at the coordinate ring of the variety S of tuples of SL_2 matrices, and see what functions in this coordinate ring are invariant under SL_2, i.e., which functions give the same value for a point $p \in S$ and its conjugates $g \cdot p$ for all $g \in SL_2$. These functions assign a value to a given orbit, and they can thus be used to tell orbits apart. The invariant functions gives rise to the algebra of invariants: an algebra consisting of all invariant functions on S. In Chapter 1, we calculate this algebra of invariants.

Next, we use geometric invariant theory. This will allow us to actually construct a "quotient" that assigns a geometric meaning to the notation (0.1.2) in terms of morphisms to some “quotient space” from a projective space with SL_2^m embedded in it. First, we describe such an embedding, and then we construct a quotient for a large part of this projective space. We do this in Chapter 2.

The general question of the existence of such quotient spaces and of so-called “geometric quotients” that completely separate the orbits raises the theory of stable and semi-stable points. In Chapter 3, we find the stable and semi-stable points in our case and see what this means for our constructed good quotient. Also, we compare our findings on (semi-)stability of points to the results on the algebra of invariants found earlier. This is done in Chapter 3.

Finally, we will consider the action of SL_2 on our space in the more general setting of representation theory. We describe how the action of SL_2 looks like as a representation, and see how we can apply Geometric Invariant Theory in this general setting. This is done in Chapter 4.

0.3. Basics of invariant theory

The main sources of the definitions in this section are [**13**], [**16**], [**6**, pp.30-52] and [**7**].

0.3.1. Coordinate ring. Let k be an infinite field, and W a finite-dimensional k-vector space. A function $f : W \to k$ is called *polynomial* if it is given by a polynomial in the coordinates with respect to a basis of W. Since a basis transformation causes a linear transformation of the basis vectors, this definition is independent of what basis we choose.

Now, by $k[W]$ we denote the k-algebra of polynomial functions on W; we call this the *coordinate ring* of W. If $w_1, ..., w_n$ is a basis of W and $x_1, ..., x_n$ is the dual basis of W^*, i.e., the coordinate functions, then $k[W] = k[x_1, ..., x_n]$.

This definition can be extended to the case where we have a coordinate ring not of a vector space, but of a general (affine) variety. Suppose we have an affine variety V defined as the zero set of a set of polynomials, $S \subset k[x_1, ..., x_n]$:

$$V = Z(S) = \{x \in \mathbb{A}^n \mid f(x) = 0 \ \forall f \in S\}.$$

Then define the ideal of all functions vanishing on V:

$$I(V) = \{f \in k[x_1, ..., x_n] \mid f(x) = 0 \ \forall x \in V\}.$$

The coordinate ring of the variety V, denoted $k[V]$, is then the quotient

$$k[V] = k[x_1, ..., x_n]/I(V).$$

For n-dimensional projective space, the coordinate ring is $k[x_1, ..., x_{n+1}]$ corresponding to the projective coordinates of the space. (Note that the elements of the coordinate ring in this case are not well-defined functions on the projective space.) Now for a projective variety V to be well-defined, the polynomials defining the variety need to be homogeneous, so the coordinate ring becomes $k[x_1, ..., x_{n+1}]/I(V)$, with $I(V)$ a homogeneous ideal.

0.3.2. The Zariski topology.

One can define a topology called the *Zariski topology* on an affine space or variety. In this topology, the closed sets are defined to be the zero sets of subsets $S \subset k[x_1, ..., x_n]$:

$$V = Z(S) = \{x \in \mathbb{A}^n \mid f(x) = 0 \ \forall f \in S\}.$$

Since $Z(V) = Z(I(V))$, and every ideal in $k[x_1, ..., x_n]$ is finitely generated, without loss of assumption we can assume that the defining set S is finite. If we have a variety V, we can define its Zariski topology as the subspace topology of the affine space it is embedded in, or, equivalently, we can define its closed sets as the zero sets of subsets $S \subset k[x_1, ..., x_n]/I(V)$. The following two lemmas indicate how the Zariski topology can be used:

LEMMA 1. *Given a variety V and non-zero $f \in k[V]$. Then the set $A = \{x \in V \mid f(x) \neq 0\}$ is Zariski-dense in V.*

PROOF. Suppose the set A was contained in some closed subset $V(S) \subset V$ and let $g \in S \subset k[V]$. Then clearly $fg = 0$. But since f is non-zero, then g must be the zero function. So in fact we must have $B = V$. □

LEMMA 2. *Given a variety V and a subset $A \subset V$ Zariski-dense in V. Suppose that for some $p \in k[V]$, $p(a) = 0$ for all $a \in A$. Then $p = 0$.*

PROOF. Suppose p is not the zero function, then $A \subset V(p) \subsetneq V$, contradicting the Zariski-denseness of A. □

The Zariski topology on a projective space or variety V is simply defined by taking the subspace topology for $V \subset \mathbb{P}^n$ as a subspace of $\mathbb{A}^{n+1}$. The same description of closed sets applies as above, except that the subsets $S \subset k[x_1, ..., x_{n+1}]$ need to consist of homogeneous polynomials.

0.3.3. Representations, modules.

Let W be a n-dimensional vector space over k. Suppose we have a group G and a group homomorphism $\rho : G \to GL(W) \cong GL_n(k)$, then ρ is called a *representation (of dimension n) of G on W*. This induces a linear group action of G on W via $g \cdot w = \rho(g)w$; W is then said to be a kG-module. Conversely, if we have a kG-module (that is, a linear group action of G on W), then this gives a representation by letting $\rho(g)$ be the map $w \mapsto g \cdot w$. Thus, we can use the terms "representation of G" and "kG-module" interchangeably.

Given kG-representations V, W of dimension m, n, their direct sum $V \oplus W$ and tensor product $V \otimes W$ are naturally kG-representations of dimension $m+n$, mn by the group actions

$$g \cdot (v, w) = (g \cdot v, g \cdot w); \ \ g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w).$$

A *morphism* between representations is a is a linear map $\phi : V \to W$ such that $g \cdot \phi(x) = \phi(g \cdot x)$ for all $g \in G$, $x \in V$. Two representations V, W of G are *equivalent* if this morphism is an isomorphism.

0.3.4. Invariants. Suppose W is a G-representation. A function $f \in k[W]$ is called *G-invariant* if $f(g \cdot w) = f(w)$ for all $g \in G$, $w \in W$. The invariant functions form a subalgebra of the coordinate ring of W which is called the *algebra of invariants*. We denote this by $k[W]^G$ (where the action of G on W is understood implicitly).

We can also phrase this differently: the G-action on W induces a G-action on $k[W]$ and $f \in k[W]$ is G-invariant if $g \cdot f = f$.

Example 3. [**7**, p.3] Let $G = k^*$, $W = k^2$ with coordinate ring $k[W] = [x, y]$, $\rho(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$. Then clearly the invariant functions are linear combinations of elements of the form $x^k y^k$: $k[W]^{k^*} = k[xy]$.

Now suppose that in addition we have an affine variety defined by some polynomials $S \subset k[x_1, .., x_n]$:

$$V = Z(S) = \{x \in \mathbb{A}^n \mid f(x) = 0 \ \forall f \in S\}.$$

Suppose, also, that V is closed under G (i.e., $g \cdot v \in V$ for all $g \in G$, $v \in V$). A function $f \in k[V] = k[x_1, ..., x_n]/I(V)$ is then similarly called G-invariant if $f(gv) = f(v)$ for all $g \in G$, $v \in V$.

The equivalent formulation also holds here: the G-action on V induces a G-action on $k[V]$ by $g \cdot (f + I(V)) = g \cdot f + I(V)$ (one can easily check well-definedness), and then $f \in k[V]$ is called invariant if $g \cdot f = f$ for any $g \in G$.

The field of invariant theory, then, is concerned with finding this algebra of invariants for a given group and variety. A result in which the generators of this structure are given, is called a *first fundamental theorem* for the algebra of invariants; a result in which the defining relations are described is called a *second fundamental theorem* (if there are none, then the algebra is called *algebraically independent*).

0.4. Notation

In our case, we are interested in the representation we get by simultaneously conjugating tuples of matrices. In the remainder of this text, let $M_{2\times 2}$ denote the 4-dimensional vector space over $\mathbb{C}$ of 2×2 matrices. Also, let M_0 be its 3-dimensional subspace of traceless matrices, and let SL_2 be the following subvariety of $M_{2\times 2}$:

$$SL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2} \mid ad - bc = 1 \right\}.$$

Consider the action of SL_2 on $M_{2\times 2}$ by conjugation (which we will denote by $\diamond$): $g \diamond M = gMg^{-1}$, where the multiplication on the right hand side is just matrix multiplication. Note that both SL_2 and M_0 are closed under this action.

We are interested in tuples of matrices in SL_2; we write

$$SL_2^k = SL_2 \times ... \times SL_2 = \{(M_1, ..., M_k) \mid M_i \in SL_2\}.$$

The action of SL_2 then extends to these tuples by simultaneous conjugation; for instance, on SL_2^k the action becomes

$$g \diamond (M_1, ..., M_k) = (gM_1g^{-1}, ..., gM_kg^{-1}).$$

When we write down $k[V]^{SL_2}$ for some coordinate ring coming from a tuple of matrices, it is understood that the action of SL_2 on V is by simultaneous conjugation. In this notation, the object we will study with invariant theory can be written as $\mathbb{C}[SL_2^m]^{SL_2}$.

0.5. Personal remarks

For their help in writing this thesis, I would like to thank: Jaap Top, my first supervisor, for suggesting this topic to me and helping me write the thesis; Marius van der Put, who gave the motivation for the problem and who helped with several suggestions, and Holger Waalkens, my second supervisor.

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CHAPTER 1

Invariant theory

In [**2**], a complete description of $\mathbb{C}[M_{2\times 2}^m]^{SL_2}$ for any k by means of first and second fundamental theorems is given.

In this chapter, we adapt the methods used in [**2**] to obtain first and second fundamental theorems for $\mathbb{C}[SL_2^m]^{SL_2}$. First, in section 1.1, we indicate how we can obtain descriptions of both $\mathbb{C}[SL_2^m]^{SL_2}$ and $\mathbb{C}[M_{2\times 2}^m]^{SL_2}$ by first determining $\mathbb{C}[M_0^m]^{SL_2}$.

Next, in section 1.2, we do this for the case $m = 2$: by elementary means we can calculate $\mathbb{C}[M_0 \times M_0]^{SL_2}$ and then determine $\mathbb{C}[M_{2\times 2} \times M_{2\times 2}]^{SL_2}$ and $\mathbb{C}[SL_2 \times SL_2]^{SL_2}$.

For the case $m \geq 3$ we can use classical results on invariants of SO_3 matrices as in [**2**]. In section 1.3 we give these classical results and derive descriptions of $\mathbb{C}[M_0^m]^{SL_2}$, $\mathbb{C}[M_{2\times 2}^m]^{SL_2}$ and $\mathbb{C}[SL_2^m]^{SL_2}$.

1.1. Traceless matrices

A standard trick that is often applied in the invariant theory of matrices as SL_2-representations is to write $M_{2\times 2} \cong M_0 \oplus \mathbb{C}$ as a sum of vector spaces. Now note that since SL_2 acts trivially on $\lambda\cdot\mathrm{Id}$, we can make $M_0 \oplus \mathbb{C}$ a SL_2-representation by

$$g \diamond (M, t) = (gMg^{-1}, t)$$

and this will make $M^0 \oplus \mathbb{C}$ isomorphic to $M_{2\times 2}$:

Lemma 4. *Under the following isomorphism ϕ, one has $M_{2\times 2} \cong M_0 \oplus \mathbb{C}$ as SL_2-representations:*

$$M \overset{\phi}{\mapsto} (M - \frac{TrM}{2} \cdot I, TrM). \tag{1.1.1}$$

Proof. We just need to show that the following diagram commutes:

$$\begin{tikzcd} M_{2\times 2} \arrow[r, "\phi"] \arrow[d, "\alpha\diamond"] & M_0 \oplus \mathbb{C} \arrow[d, "\alpha\diamond"] \\ M_{2\times 2} \arrow[r, "\phi"] & M_0 \oplus \mathbb{C}. \end{tikzcd}$$

But this is easily seen because

$$\begin{aligned} \phi(\alpha \diamond M) &= \left(\alpha M\alpha^{-1} - \frac{\mathrm{Tr}\left(\alpha M \alpha^{-1}\right)}{2} \cdot I, \mathrm{Tr}\left(\alpha M\alpha^{-1}\right)\right) \\ &= \left(\alpha\left(M - \frac{\mathrm{Tr}M}{2} \cdot I\right)\alpha^{-1}, \mathrm{Tr}M\right) \\ &= \alpha \diamond \left(M - \frac{\mathrm{Tr}M}{2} \cdot I, \mathrm{Tr}M\right) = \alpha \diamond \phi(M). \end{aligned}$$

□

Let us now consider the coordinate ring $\mathbb{C}[M_0 \oplus \mathbb{C}] = \mathbb{C}[M_0] \otimes \mathbb{C}[t]$. Note that SL_2 acts on $\mathbb{C}[M_0]$ in the usual way, and it acts trivially on $\mathbb{C}[t]$. But then clearly we must have

$$\mathbb{C}[M_0 \oplus \mathbb{C}]^{SL_2} = (\mathbb{C}[M_0] \otimes \mathbb{C}[t])^{SL_2} = \mathbb{C}[M_0]^{SL_2} \otimes \mathbb{C}[t].$$

Clearly the isomorphism ϕ also induces an isomorphism between the coordinate rings $\mathbb{C}[M_0 \oplus \mathbb{C}]$ and $\mathbb{C}[M_{2\times 2}]$, so:

$$\mathbb{C}[M_{2\times 2}]^{SL_2} \cong \mathbb{C}[M_0]^{SL_2} \otimes \mathbb{C}[t]. \tag{1.1.2}$$

We can also generalize the reasoning by which we obtained (1.1.2) to tuples of matrices. Let us first introduce some notation for the coordinate rings we will use throughout this chapter.

For M_0^m, let a_i, b_i, c_i for $1 \leq i \leq m$ be the coordinate functions of this vector space forming generic matrices

$$y_i = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix}.$$

Thus, we can talk about $\mathbb{C}[M_0^m]$, $\mathbb{C}[a_1, b_1, c_1, ..., a_m, b_m, c_m]$ or $\mathbb{C}[y_1, ..., y_m]$ interchangeably.

For $M_{2\times 2}^m$, let a_i', b_i', c_i', d_i' for $1 \leq i \leq m$ be the coordinate functions forming generic matrices

$$x_i = \begin{pmatrix} a_i' & b_i' \\ c_i' & d_i' \end{pmatrix}.$$

For SL_2^m, the notation is the same as in the case of $M_{2\times 2}^m$, except that we are working in a subvariety. So the coordinate ring we consider is

$$\mathbb{C}[SL_2^m] = \mathbb{C}[x_1, ..., x_m]/(I); \quad (I) = (\det x_1 - 1, ..., \det x_m - 1).$$

So how do the above considerations generalize to tuples of matrices? Lemma 4 is obviously also true for tuples, so we have the following isomorphism:

$$\begin{array}{rcl} M_{2\times 2}^m & \to & M_0^m \oplus \mathbb{C}^{\oplus m} \\ (M_1, ..., M_m) & \mapsto & (M_1 - \frac{1}{2}\mathrm{Tr} M_1 \cdot I, ..., \mathrm{Tr} M_1, ..., \mathrm{Tr} M_m). \end{array} \tag{1.1.3}$$

This induces $\mathbb{C}[M_{2\times 2}^m] \cong \mathbb{C}[M_0^m] \otimes \mathbb{C}[t_1, ..., t_m]$, and, reasoning as above, we immediately obtain the following result:

Proposition 5. *We have an isomorphism of SL_2-modules:*

$$\mathbb{C}[M_{2\times 2}^m]^{SL_2} \cong \mathbb{C}[M_0^m]^{SL_2} \otimes \mathbb{C}[t_1, ..., t_m].$$

The identification is induced by (1.1.3).

In the SL_2 case, the result is slightly more involved:

Proposition 6. *We have an isomorphism of SL_2-modules:*

$$\mathbb{C}[SL_2^m]^{SL_2} \cong \left(\mathbb{C}[M_0^m]^{SL_2} \otimes \mathbb{C}[t_1, ..., t_m]\right)/(I'),$$

where

$$(I') = (dety_1 + \frac{1}{4}t_1^2 - 1, ..., dety_m + \frac{1}{4}t_m^2 - 1).$$

The identification is induced by (1.1.3).

Proof. $\mathbb{C}[SL_2^m]$ has the same generators of the coordinate ring as $M_{2\times 2}$, but they are divided out by the ideal

$$(I) = (\det x_1 - 1, ..., \det x_m - 1).$$

One verifies that under ϕ, this ideal gets translated to the ideal

$$(I') = (\det y_1 + \frac{1}{4}t_1^2 - 1, ..., \det y_m + \frac{1}{4}t_m^2 - 1),$$

so we have the SL_2-module isomorphism

$$\mathbb{C}[SL_2^m] \cong \left(\mathbb{C}[M_0^m] \otimes \mathbb{C}[t_1, ..., t_m]\right)/I'.$$

Now by eliminating higher powers of the t_i using the ideal, we can write

$$\mathbb{C}[SL_2^m] \cong \mathbb{C}[M_0^m] \otimes (1 \oplus t_1) \otimes ... \otimes (1 \oplus t_m),$$

and again, we note that SL_2 just acts on $\mathbb{C}[M_0^m]$, so

$$\mathbb{C}[SL_2^m]^{SL_2} \cong \mathbb{C}[M_0^m]^{SL_2} \otimes (1 \oplus t_1) \otimes ... \otimes (1 \oplus t_m).$$

Introducing the ideal (I') again we get our claim. ☐

We note some identities that will be useful later on:

LEMMA 7. *Under the isomorphism $\phi : \mathbb{C}[M_0^m] \otimes \mathbb{C}[t_1, ..., t_m] \to \mathbb{C}[M_{2\times 2}^m]$ induced by (1.1.3), we have*

$$\begin{aligned} \phi(t_i) &= \mathit{Tr} x_i \\ \phi(\mathit{Tr} y_i y_j) &= \mathit{Tr} x_i x_j - \frac{1}{2}\mathit{Tr} x_i \, \mathit{Tr} x_j \\ \phi(\mathit{Tr}(y_i y_j y_k - y_i y_k y_j)) &= \mathit{Tr}(x_i x_j x_k - x_i x_k x_j). \end{aligned}$$

PROOF. One verifies this just by direct computation, e.g.:

$$\begin{aligned} \phi(\mathrm{Tr} y_i y_j) &= \mathrm{Tr}((x_i - \frac{1}{2}\mathrm{Tr} x_i \cdot I) \cdot (x_j - \frac{1}{2}\mathrm{Tr} x_j \cdot I) \\ &= \mathrm{Tr}(x_i x_j - \frac{1}{2}\mathrm{Tr}\,(x_j) \cdot x_i - \frac{1}{2}\mathrm{Tr}\,(x_i)\, x_j + \frac{1}{4}\mathrm{Tr} x_i \mathrm{Tr} x_j \cdot I) \\ &= \mathrm{Tr} x_i x_j - \frac{1}{2}\mathrm{Tr} x_i \mathrm{Tr} x_j. \end{aligned}$$

☐

1.2. Two 2×2 matrices

Now, we will calculate the algebra of invariants for $M_0 \times M_0$. This proof is a more detailed version of the one found in [**7**, p.21]. We will prove the following:

PROPOSITION 8. $\mathbb{C}[M_0 \times M_0]^{SL_2} = \mathbb{C}[\mathit{Tr} y_1 y_2, \mathit{Tr} y_1^2, \mathit{Tr} y_2^2]$, *and these generators are algebraically independent.*

PROOF. First of all, we prove algebraic independence. Consider the following evaluation homomorphism:

$$\begin{aligned} \mathbb{C}[a_1, b_1, c_1, a_2, b_2, c_2] &\overset{\psi}{\to} \mathbb{C}[\nu, \mu, \lambda] \\ a_1 &\mapsto \nu \\ b_1 &\mapsto \mu \\ a_2 &\mapsto \lambda \\ c_1, b_2, c_2 &\mapsto 0. \end{aligned} \tag{1.2.1}$$

Note that under this homomorphism, $\mathrm{Tr}\ y_1^2 \mapsto 2\nu^2$; $\mathrm{Tr} y_2^2 \mapsto 2\lambda$; $\mathrm{Tr} y_1 y_2 \mapsto \mu$. Suppose there was was a non-trivial relation between $\mathrm{Tr} A^2$, $\mathrm{Tr} B^2$, $\mathrm{Tr} AB$: then this would translate under ψ to a non-trivial relation between $2\nu^2$, 2λ, μ. But these three elements are clearly algebraically independent in $\mathbb{C}[\nu, \mu, \lambda]$, giving a contradiction.

Next we prove that any invariant function is a polynomial in these three traces. So let $F \in \mathbb{C}[a_1, b_1, c_1, a_2, b_2, c_2]$ be an invariant function. Let us consider the following evaluation homomorphism:

$$\begin{array}{rcl} \mathbb{C}[a_1, b_1, c_1, a_2, b_2, c_2] & \xrightarrow{\phi} & \mathbb{C}[a, c, t] \\ a_1 & \mapsto & t \\ b_1, c_1 & \mapsto & 0 \\ a_2 & \mapsto & a \\ b_2 & \mapsto & 1 \\ c_2 & \mapsto & c. \end{array}$$

Hence we get $\phi(F) \in \mathbb{C}[a, c, t]$. But now notice that for any $a, c \neq 0, t \neq 0 \in \mathbb{C}$, we have

$$\begin{pmatrix} 0 & 1/\sqrt{c} \\ \sqrt{c} & 0 \end{pmatrix} \diamond \left(\begin{pmatrix} t & \\ & -t \end{pmatrix}, \begin{pmatrix} a & 1 \\ c & -a \end{pmatrix} \right) = \left(\begin{pmatrix} -t & \\ & t \end{pmatrix}, \begin{pmatrix} -a & 1 \\ c & a \end{pmatrix} \right).$$

Because F is invariant under conjugation, evaluation of $\phi(F)$ in the second matrix pair must give the same result as evaluating in the first matrix pair, ie, we have

$$\phi(F)(a, c, t) = \phi(F)(-a, c, -t)$$

in the ring $\mathbb{C}[a, c, t]$. Hence $\phi(F)$ is a polynomial in a^2, at, t^2 and c. But we note the following identities:

$$t^2 = \phi(\frac{1}{2}\mathrm{Tr} y_1^2); \quad a^2 + c = \phi(\frac{1}{2}\mathrm{Tr} y_2^2); \quad at = \phi(\frac{1}{2}\mathrm{Tr} y_1 y_2);$$
$$\phi(\mathrm{Tr} y_1^2) c = \phi(\frac{1}{2}\mathrm{Tr} y_1^2 \mathrm{Tr} y_2^2 - (\mathrm{Tr} y_1 y_2)^2),$$

so that for certain $k \geq 0$ and $P \in \mathbb{C}[\mathrm{Tr} y_1^2, \mathrm{Tr} y_2^2, \mathrm{Tr} y_1 y_2]$ we have

$$\phi((\mathrm{Tr} y_1^2)^k) \cdot \phi(F) = \phi(P). \tag{1.2.2}$$

We need the fact that the set

$$\begin{aligned} M' \quad = \quad & \{(A, B) \in M_0^2 \mid \exists Q \in SL_2, a, c \neq 0, t \neq 0 \in \mathbb{C} : \\ & Q(A, B)Q^{-1} = \left(\left(\begin{smallmatrix} t & \\ & -t \end{smallmatrix} \right), \left(\begin{smallmatrix} a & 1 \\ c & -a \end{smallmatrix} \right) \right) \} \end{aligned}$$

is Zariski-dense in M_0^2. We prove this rather technical fact in Lemma 9 below. Now take any $p = (A, B) \in M'$, then p is conjugate to a point $p' = \left(\left(\begin{smallmatrix} t & 0 \\ 0 & -t \end{smallmatrix} \right), \left(\begin{smallmatrix} a & 1 \\ c & -a \end{smallmatrix} \right) \right) \in M'$ in diagonal form, so:

$$\begin{array}{rcll} ((\mathrm{Tr} A^2)^k F)(p) & = & ((\mathrm{Tr} A^2)^k F)(p') & \text{(invariance of } F\text{)} \\ & = & \phi((\mathrm{Tr} A^2)^k)\phi(F)(a, c, t) & \text{(definition of } \phi\text{)} \\ & = & \phi(P)(a, c, t) & \text{(1.2.2)} \\ & = & P(p') & \text{(definition of } \phi\text{)} \\ & = & P(p) & \text{(invariance of } P\text{),} \end{array}$$

but then by Lemma 2, the Zariski-denseness of M' implies that this holds as a polynomial equation on the whole of M_0^2:

$$(\mathrm{Tr} y_1^2)^k F = P. \tag{1.2.3}$$

Thus, so far, we have established that

$$F \in \mathbb{C}[\mathrm{Tr} y_1^2, \mathrm{Tr} y_2^2, \mathrm{Tr} y_1 y_2, \frac{1}{\mathrm{Tr} y_1^2}] \cap \mathbb{C}[a_1, b_1, c_1, a_2, b_2, c_2].$$

Now on the one hand, F is just a polynomial 6 variables; on the other hand, it can be a fraction in $\mathrm{Tr} y_1^{-2}$ which would have some asymptotic behaviour. We would like to conclude that in fact this fraction cannot occur, so we have a polynomial $F \in \mathbb{C}[\mathrm{Tr} y_1^2, \mathrm{Tr} y_2^2, \mathrm{Tr} y_1 y_2]$, finishing our proof.

First, write F as a Laurent polynomial in $\mathrm{Tr}y_1^2$:

$$F = \sum_{i=i_0}^{k} f_i \cdot \left(\mathrm{Tr}y_1^2\right)^i;\ f_i \in \mathbb{C}[\mathrm{Tr}y_1y_2, \mathrm{Tr}y_2^2],$$

and suppose that i_0 is negative. If we apply the evaluation homomorphism ψ from (1.2.1) to F, we get

$$\psi(F) = \sum_{i=i_0}^{k} f'_i \cdot \left(2\nu^2\right)^i \in \mathbb{C}[\mu, \lambda](\nu);\ \ f'_i = \psi(f_i) \in \mathbb{C}[\mu, \lambda].$$

Now, since f_{i_0} is not the zero function, we can choose $\hat{\mu}$, $\hat{\lambda}$ so that $f_{i_0}(\hat{\mu}, \hat{\lambda}) \neq 0$. Now apply the evaluation homomorphism $\chi:\ \mu \mapsto \hat{\mu};\ \lambda \mapsto \hat{\lambda}$, then

$$\chi(\psi(F)) = \sum_{i=i_0}^{k} f'_i(\hat{\mu}, \hat{\lambda}) \cdot \left(2\nu^2\right)^k \in \mathbb{C}(\nu),$$

so we have constructed a function in $\chi(\psi(F)) \in \mathbb{C}(\nu)$ which by construction tends to $\pm\infty$ as ν tends to 0.

However, we also had $F \in \mathbb{C}[a_1, b_1, c_1, a_2, b_2, c_2]$, so in fact $\psi(F) \in \mathbb{C}[\nu, \mu, \lambda]$ and $\chi(\psi(F)) \in \mathbb{C}[\nu]$! This contradicts the asymptotic behaviour that we just established. So in fact, i_0 cannot be negative, and $F \in \mathbb{C}[\mathrm{Tr}y_1^2, \mathrm{Tr}y_2^2, \mathrm{Tr}y_1y_2]$ as we wanted to show. □

We are just left with the technical lemma we had:

LEMMA 9. *The set*

$$\begin{aligned} S &= \{(A,B) \in M_0^2 \mid \exists Q \in SL_2, a, c \neq 0, t \neq 0 \in \mathbb{C}: \\ &\qquad Q(A,B)Q^{-1} = \left(\left(\begin{smallmatrix} t & \\ & -t \end{smallmatrix}\right), \left(\begin{smallmatrix} a & 1 \\ c & -a \end{smallmatrix}\right)\right)\} \end{aligned}$$

is Zariski-dense in M_0^2.

PROOF. First let

$$S' = \{(A,B) \in M_0^2 \mid A \text{ has two distinct eigenvalues}\}.$$

It is standard that this set is Zariski-dense in S: as a defining equation for Lemma 1 we can use the fact that the discriminant of its characteristic polynomial must be non-zero.

Now we consider the subset

$$S'' = \{(A,B) \in S' \mid \mathrm{Tr}A^2\mathrm{Tr}B^2 - (\mathrm{Tr}AB)^2 \neq 0\}.$$

This set is obviously Zariski-dense in S' and hence in S; we claim that this is the set we need.

For let us consider a tuple (A, B) where A has distinct eigenvalues. Then for some matrix Q,

$$Q(A,B)Q^{-1} = \left(\begin{pmatrix} t & \\ & -t \end{pmatrix}, \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right);\ t \neq 0.$$

Now,

$$\begin{aligned} \mathrm{Tr}A^2\mathrm{Tr}B^2 - (\mathrm{Tr}AB)^2 &= \left(2t^2\right)\left(2a^2 + 2bc\right) - (2at)^2 \\ &= 4a^2t^2 + 4t^2bc - 4a^2t^2 = 4t^2bc. \end{aligned}$$

Then for this to be non-zero, we need $b, c \neq 0$.

But now we note that in choosing Q to diagonalize A, we have a degree of freedom that enables us to multiply b with a non-zero constant μ^2; c then gets multiplied with μ^{-2}. So in fact we can make $b = 1$, $c \neq 0$, so $(A, B) \in S$, and we are done. □

But now Propositions 5 and 6 allow us to translate these results to $\mathbb{C}[M_{2\times2}^2]^{SL_2}$ and $\mathbb{C}[SL_2^2]^{SL_2}$:

Corollary 10. $\mathbb{C}[M_{2\times2}\times M_{2\times2}]^{SL_2}=\mathbb{C}[\mathit{Tr}x_1, \mathit{Tr}x_2, \mathit{Tr}x_1x_2, \mathit{Tr}x_1^2, \mathit{Tr}x_2^2]$, *and these generators are algebraically independent.*

Proof. By Proposition 5,

$$\begin{aligned}\mathbb{C}[M_{2\times2}\times M_{2\times2}]^{SL_2} &\cong \mathbb{C}[M_0^2]^{SL_2}\otimes\mathbb{C}[t_1,t_2].\\ &= \mathbb{C}[\mathrm{Tr}y_1y_2, \mathrm{Tr}y_1^2, \mathrm{Tr}y_2^2]\otimes\mathbb{C}[t_1,t_2].\end{aligned}$$

Now, translating this ring under the isomorphism (1.1.3) and using Lemma 4, we get the required result. ☐

Corollary 11. $\mathbb{C}[SL_2\times SL_2]^{SL_2}=\mathbb{C}[\mathit{Tr}x_1, \mathit{Tr}x_2, \mathit{Tr}x_1x_2]$, *and these generators are algebraically independent.*

Proof. By Proposition 6, letting $(I')=(\det y_1+\frac{1}{4}t_1^2-1, \det y_2+\frac{1}{4}t_2^2-1)$, we have:

$$\begin{aligned}\mathbb{C}[SL_2^2]^{SL_2} &\cong \left(\mathbb{C}[M_0^2]^{SL_2}\otimes\mathbb{C}[t_1]\otimes\mathbb{C}[t_2]\right)/(I'),\\ &= \left(\mathbb{C}[\mathrm{Tr}y_1y_2, \mathrm{Tr}y_1^2, \mathrm{Tr}y_2^2]\otimes\mathbb{C}[t_1]\otimes\mathbb{C}[t_2]\right)/(I'),\\ &\cong \mathbb{C}[\mathrm{Tr}y_1y_2, t_1, t_2],\end{aligned}$$

where for the last isomorphism we eliminated $\mathrm{Tr}y_i^2$ by noting the following equality in $\left(\mathbb{C}[\mathrm{Tr}y_1y_2, \mathrm{Tr}y_1^2, \mathrm{Tr}y_2^2]\otimes\mathbb{C}[t_1]\otimes\mathbb{C}[t_2]\right)/(I')$:

$$\mathrm{Tr}y_i^2=-2\det y_i=\frac{1}{2}t_i^2-2.$$

Using the isomorphism (1.1.3) and Lemma 4, we get the required result. ☐

1.3. The general case

We now calculate $\mathbb{C}[M_0^k]^{SL_2}$ for any $k\geq 2$. We do this by using classical results on invariants of SO_3-representations.

First we introduce these classical results; in the second subsection, we use them to determine $\mathbb{C}[M_0^k]^{SL_2}$. The approach taken here is a slight adaptation from the the one presented in [**2**]. In the third subsection, we apply these results, much like before, to find $\mathbb{C}[M_{2\times2}^k]^{SL_2}$ and $\mathbb{C}[SL_2^k]^{SL_2}$.

1.3.1. Invariants of orthorgonal groups.

Let V_n be a n-dimensional vector space with basis $\{e_1, ..., e_n\}$. Define a non-degenerate, symmetric bilinear form called the *dot product* by:

$$\left\langle\sum v_ie_i, \sum w_ie_i\right\rangle=\sum v_iw_i.$$

Let O_n be the group of matrices $g\in GL(V_n)$ so that $\langle g\cdot x, g\cdot x\rangle=\langle x, x\rangle$ for all $x\in V_n$; this is well known to be equivalent to the matrix satisfying $G^TG=I$. These matrices have determinant ±1; the subgroup SO_n consists of the matrices in O_n with determinant 1.

Now for vectors $v_i=(v_{i1}, .., v_{in})$ and $w_i=(w_{i1}, .., w_{in})$, we define

$$\Delta_n(v_1, v_2, .., v_n)=\det(v_1, .., v_n)=\begin{vmatrix} v_{11} & v_{21} & \cdots & v_{n1}\\ v_{12} & & & v_{n2}\\ \vdots & & & \vdots\\ v_{1n} & v_{2n} & \cdots & v_{nn}\end{vmatrix};$$

$$\Gamma_k(v_1, ..., v_k | w_1, ..., w_k) = \det\left(\langle v_i, w_j \rangle |_{i,j=1,...,k} \right) = \begin{vmatrix} \langle v_1, w_1 \rangle & \cdots & \langle v_1, w_k \rangle \\ \langle v_2, w_1 \rangle & & \langle v_2, w_k \rangle \\ \vdots & & \vdots \\ \langle v_k, w_1 \rangle & \cdots & \langle v_k, w_k \rangle \end{vmatrix}.$$

The vector space V_n^m is a SO_n- or O_n-representation by letting the groups act diagonally on m copies of V_n by matrix multiplication, and the first and second fundamental theorems of invariant theory for $\mathbb{C}[V_n^m]^{O_n}$ and $\mathbb{C}[V_n^m]^{SO_n}$ are well-known; for example, in his classical work [**12**], Weyl gives both the First Fundamental Theorem ([**12**, p.53]), and the Second Fundamental Theorem ([**12**, p.75]), albeit he works only over the real numbers. In [**2**], we find them stated as follows.

We introduce the following notation: let $v_{i,j}$ be the coordinate functions for the ith copy of V_n in V_n^m.

THEOREM 12. *(The fundamental theorems for $\mathbb{C}[V_n^m]^{O_n}$) Consider $\mathbb{C}[V_n^m]^{O_n}$.*

(i) Its algebra of invariants is generated by $\langle v_i, v_j \rangle$ for $1 \leq i, j \leq m$;

(ii) the defining relations for the algebra of invariants are ($1 \leq i_0 < i_1 < ... < i_n \leq m$; $1 \leq j_0 < j_1 < ... < j_n \leq m$):

$$\Gamma_{n+1}(v_{i_0}, v_{i_1}, ..., v_{i_n} | v_{j_0}, v_{j_1}, ..., v_{j_n}) = 0.$$

Similarly for SO_n:

THEOREM 13. *(The fundamental theorems for $\mathbb{C}[V_n^m]^{SO_n}$) Consider $\mathbb{C}[V_n^m]^{SO_n}$.*

(i) Its algebra of invariants is generated by the generators of $\mathbb{C}[V_n^m]^{O_n}$ and $\Delta(v_{i_1}, v_{i_2}, ..., v_{i_n})\mathbb{C}[V_n^m]^{O_n}$, $1 \leq i_1 < ... < i_n \leq m$;

(ii) the defining relations for the algebra of invariants are

$$\Delta_n(u_{i_1}, ..., u_{i_n})\Delta_n(u_{j_1}, ..., u_{j_n}) - \Gamma_n(u_{i_1}, ..., u_{i_n} | u_{j_1}, ..., u_{j_n}) = 0$$

$$(1 \leq i_1 < ... < i_n < m; 1 \leq j_1 < ... < j_n \leq m);$$

$$\sum_{k=0}^{n} (-1)^k \langle u_i, u_{j_k} \rangle \Delta_n(u_{j_0}, ..., \widehat{u_{j_k}}, ..., u_{j_n}) = 0$$

$$1 \leq j_0 < j_1 < ... < j_n \leq m$$

(here, $\widehat{u_{j_k}}$ means that this u_{j_k} is removed from the expression).

Note that for SO_n we do not also need the defining relations for O_n: it turns out that these relations already follow from the relations mentioned in the Theorem for SO_n.

1.3.2. Connecting the actions of SO_3 and SL_2. In section 1.3.1, we considered the invariant functions on V_n^m under the action of SO_n.

Let us now look at the the case of SO_3 acting on V_3^m. We can identify this vector space with the space M_0^m, and fix basis

$$\left\{ e_{i,1} = \frac{1}{2}\sqrt{2} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right), e_{i,2} = \frac{1}{2}\sqrt{2} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right), e_{i,3} = \frac{1}{2}\sqrt{-2} \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \right\}, \ 1 \leq i \leq m;$$

one can now check that for the bilinear form we have $\langle u, v \rangle = \mathrm{Tr} uv$.

Our representation of SL_2 on M_0^m by simultaneous conjugation is simply a group homomorphism $\phi : SL_2 \to GL(W)$. We claim that this map is a surjection to SO_3. But then $\mathbb{C}[M_0^m]^{SL_2} = \mathbb{C}[V_3^m]^{SO_3}$, and we just have to translate the invariant theory of SO_3 to the SL_2 case. First our claim:

LEMMA 14. *The mapping ϕ above is a surjective map $SL_2 \to SO_3$.*

Proof. Note that we know that the action of SL_2 (which we denote by $\diamond$) preserves the dot product, because the dot product of u and v is the same as the trace of uv, which is well-known to be invariant under conjugation. So $\phi(SL_2) \subset O_3$.

We will show the following intermediate result:

$$\forall M \in SO_3\ \exists Q \in SL_2 :\ Q \diamond e_1 = Me_1,\ Q \diamond e_2 = Me_2. \tag{1.3.1}$$

For suppose we have shown (1.3.1). Denote $f_i = Me_i$, and note that because the dot product is preserved under the action of M, we have $\langle f_i, f_j\rangle = \delta_{ij}$. Then:

$$\begin{aligned} Q \diamond e_3 &= \langle Q \diamond e_3, f_1\rangle f_1 + \langle Q \diamond e_3, f_2\rangle f_2 + \langle Q \diamond e_3, f_3\rangle f_3 \\ &= \langle Q \diamond e_3, Q \diamond e_1\rangle f_1 + \langle Q \diamond e_3, Q \diamond e_2\rangle f_2 + \langle Q \diamond e_3, f_3\rangle f_3 \\ &= \langle e_3, e_1\rangle f_1 + \langle e_3, e_2\rangle f_2 + \langle Q \diamond e_3, f_3\rangle f_3 = \langle Q \diamond e_3, f_3\rangle f_3. \end{aligned}$$

But also:

$$1 = \langle e_3, e_3\rangle = \langle Q \diamond e_3, Q \diamond e_3\rangle = \langle Q \diamond e_3, f_3\rangle^2 .$$

So then either $\phi(Q) = M \in SO_3$, or $\phi(Q) = M' \in O_3 \backslash SO_3$, where M' is M but with the last row multiplied by -1. We then show that in fact $\det\phi(Q) = 1$ holds for any $Q \in SL_2$, which will conclude our proof.

Now we prove the statement (1.3.1). Since by assumption, $\langle f_1, f_1\rangle = 1$, we have that f_1 must be diagonalizable. For otherwise conjugating it into Jordan form gives one of the non-diagonalizable matrices, which have norm 0:

$$\left\langle \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \right\rangle = 0, \text{ or } \langle 0, 0\rangle = 0.$$

So the Jordan form of f_1 has to be

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \text{ for some } \lambda \neq 0,$$

and then preservation of the norm gives us that in fact f_1 must diagonalize to e_1, ie, $Q^{-1} f_1 Q = e_1$ for some $Q \in SL_2$; thus, $Q^{-1} \diamond e_1 = f_1$.

Now, look at the action of Q on f_2: let's suppose $Q \diamond f_2 = ae_2 + be_3$ (we have $\langle Q \diamond f_2, e_1\rangle = 0$ by the above). Note that we can always find α such that

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \diamond e_2 = ae_2 + be_3 :$$

a calculation shows that this conjugation is equal to $\frac{1}{2}((\alpha^2 + \alpha^{-2})e_2 - i(\alpha^2 - \alpha^{-2})e_3)$, so one sees that choosing $\alpha = \sqrt{a - bi}$ does the trick.

But then note that

$$\left(Q^{-1} \cdot \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\right) \diamond e_2 = Q^{-1} \diamond \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \diamond e_2 = Q^{-1} \diamond (ae_2 + be_3) = f_2,$$

and, since conjugating with this diagonal matrix does not change e_1, we see that the matrix

$$Q' = Q^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in SL_2$$

has the property $Q'e_1 = f_1$, $Q'e_2 = f_2$, as requested. This proves (1.3.1).

We now show that $\det\phi(M) = 1$ for any diagonalizable $M \in SL_2$. Let us say that M can be diagonalized with matrix S to $\mathrm{diag}(\lambda, \lambda^{-1})$. Then:

$$\begin{aligned} \det\phi(M) &= \det\phi\left(S \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} S^{-1}\right) \\ &= \det\phi(S) \det\phi \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \det\phi(S)^{-1} = \det\phi \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}. \end{aligned}$$

We already saw how such a diagonal matrix acts on e_2; a calculation similarly shows that it sends e_3 to $\frac{1}{2}(-i(\lambda^2-\lambda^{-2})e_2+(\lambda^2+\lambda^{-2})e_3)$, so we get the following

$$\det\phi\begin{pmatrix}\lambda & \\ & \lambda^{-1}\end{pmatrix}=\begin{vmatrix}1 & 0 & 0\\ 0 & \frac{1}{2}(\lambda^2+\lambda^{-2}) & \frac{1}{2}i(\lambda^2-\lambda^{-2})\\ 0 & -\frac{1}{2}i(\lambda^2-\lambda^{-2}) & \frac{1}{2}(\lambda^2+\lambda^{-2})\end{vmatrix}=1.$$

We note that $\det\phi(M)$ is a homogeneous polynomial of degree 6 in the coordinate functions of M: the elements of the O_3-matrix are products of the entries of M and M^{-1} and M^{-1} does not have fractions because M has determinant 1. So, we have that $\det\phi(M)-1=0$ for all M in the set of diagonalizable matrices in SL_2.

But this set is Zariski-dense in SL_2 because its subset consisting of matrices with two different eigenvalues is already dense in SL_2 (this can be expressed in a polynomial equation by saying that the discriminant of the characteristic polynomial is nonzero, then use Lemma 1). So, in fact $\det\phi(M)=1$ for all $m\in SL_2$ by Lemma 2, completing the proof. □

1.3.3. The structure of $\mathbb{C}[M_0^m]^{SL_2}$, $\mathbb{C}[M_{2\times 2}^m]^{SL_2}$, $\mathbb{C}[SL_2^m]^{SL_2}$. Now, as argued, Theorem 13 directly gives us $\mathbb{C}[M_0^m]^{SL_2}$; we just re-formulate it to get rid of the dot product notation. As generators, we have $\langle y_i,y_j\rangle=\mathrm{Tr}y_iy_j$ and $\Delta(y_{i_1},y_{i_2},y_{i_3})$. We can also write this Δ in terms of traces, for one verifies that the following relation holds:

$$\Delta(y_{i_1},y_{i_2},y_{i_3})=-\frac{1}{\sqrt{-2}}\mathrm{Tr}(y_{i_1}y_{i_2}y_{i_3}-y_{i_1}y_{i_3}y_{i_2}). \tag{1.3.2}$$

For the first type of relation given in the theorem, we note that

$$\begin{aligned}\Gamma_k(y_{i_1},y_{i_2},y_{i_3}|y_{j_1},y_{j_2},y_{j_3}) &= \begin{vmatrix}\langle y_{i_1},y_{j_1}\rangle & \langle y_{i_1},y_{j_2}\rangle & \langle y_{i_1},y_{j_3}\rangle\\ \langle y_{i_2},y_{j_1}\rangle & \langle y_{i_2},y_{j_2}\rangle & \langle y_{i_2},y_{j_3}\rangle\\ \langle y_{i_3},y_{j_1}\rangle & \langle y_{i_3},y_{j_2}\rangle & \langle y_{i_3},y_{j_3}\rangle\end{vmatrix}\\ &= \begin{vmatrix}\mathrm{Tr}y_{i_1}y_{j_1} & \mathrm{Tr}y_{i_1}y_{j_2} & \mathrm{Tr}y_{i_1}y_{j_3}\\ \mathrm{Tr}y_{i_2},y_{j_1} & \mathrm{Tr}y_{i_2}y_{j_2} & \mathrm{Tr}y_{i_2}y_{j_3}\\ \mathrm{Tr}y_{i_3},y_{j_1} & \mathrm{Tr}y_{i_3}y_{j_2} & \mathrm{Tr}y_{i_3}y_{j_3}\end{vmatrix}.\end{aligned}$$

The second type of relation directly translates using (1.3.2). Hence, we get:

THEOREM 15. *Consider $\mathbb{C}[M^0\times\ldots\times M^0]^{SL_2}$, where SL_2 acts by simultaneous conjugation on m matrices. Then*

(i) this algebra of invariants is generated by $\mathit{Tr}y_jy_k$, $1\leq j\leq k\leq m$, and

$$s_3(y_{j_1},y_{j_2},y_{j_3}):=y_{j_1},y_{j_2},y_{j_3}-y_{j_1},y_{j_3},y_{j_2};\ 1\leq j_1<j_2<j_3\leq m;$$

(ii) the defining relations for the invariant ring are

$$\mathit{Tr}s_3(y_{i_1}y_{i_2}y_{i_3})\cdot\mathit{Tr}s_3(y_{j_1}y_{j_2}y_{j_3})+2\begin{vmatrix}\mathit{Tr}y_{i_1}y_{j_1} & \mathit{Tr}y_{i_1}y_{j_2} & \mathit{Tr}y_{i_1}y_{j_3}\\ \mathit{Tr}y_{i_2},y_{j_1} & \mathit{Tr}y_{i_2}y_{j_2} & \mathit{Tr}y_{i_2}y_{j_3}\\ \mathit{Tr}y_{i_3},y_{j_1} & \mathit{Tr}y_{i_3}y_{j_2} & \mathit{Tr}y_{i_3}y_{j_3}\end{vmatrix}=0;$$

$1\leq i_1<i_2<i_3\leq m$; $1\leq j_1<j_2<j_3\leq m$; and

$$\sum_{k=0}^{3}(-1)^k\,\mathit{Tr}y_iy_{p_k}\,\mathit{Tr}s_3(y_{p_0},\ldots,\widehat{y_{p_k}},\ldots,y_{p_3})=0;$$

$1\leq p_0<p_1<p_2<p_3\leq m$. Here, $\hat{x}$ means x is removed from the expression.

As before, a rather straightforward application of Proposition 5 and Lemma 7 now directly gives us:

COROLLARY 16. *Consider* $\mathbb{C}[M_{2\times 2} \times ... \times M_{2\times 2}]^{SL_2}$*, where* SL_2 *acts by simultaneous conjugation on* m *matrices. Then*

(i) this algebra of invariants is generated by $\mathit{Tr}x_j$*,* $1 \leq j \leq m$*,* $\mathit{Tr}x_j x_k$*,*$1 \leq j \leq k \leq m$*, and*

$$s_3(x_{j_1}, x_{j_2}, x_{j_3}) := x_{j_1}, x_{j_2}, x_{j_3} - x_{j_1}, x_{j_3}, x_{j_2};\ 1 \leq j_1 < j_2 < j_3 \leq m;$$

(ii) the defining relations for the invariant ring are

$$\mathit{Tr}s_3(x_{i_1}x_{i_2}x_{i_3}) \cdot \mathit{Tr}s_3(x_{j_1}x_{j_2}x_{j_3}) + \\ 2\begin{vmatrix} \mathit{Tr}x_{i_1}x_{j_1} - \frac{1}{2}\mathit{Tr}x_{i_1}\,\mathit{Tr}x_{j_1} & \cdots & \mathit{Tr}x_{i_1}x_{j_3} - \frac{1}{2}\mathit{Tr}x_{i_1}\,\mathit{Tr}x_{j_3} \\ \vdots & \ddots & \vdots \\ \mathit{Tr}x_{i_3}x_{j_1} - \frac{1}{2}\mathit{Tr}x_{i_3}\,\mathit{Tr}x_{j_1} & \cdots & \mathit{Tr}x_{i_3}x_{j_3} - \frac{1}{2}\mathit{Tr}x_{i_3}\,\mathit{Tr}x_{j_3} \end{vmatrix} = 0;$$

$1 \leq i_1 < i_2 < i_3 \leq m$*;* $1 \leq j_1 < j_2 < j_3 \leq m$*; and*

$$\sum_{k=0}^{3}(-1)^k \left(\mathit{Tr}x_i x_{p_k} - \frac{1}{2}\mathit{Tr}x_i\,\mathit{Tr}x_{p_k}\right)\mathit{Tr}s_3(y_{p_0}, ..., \widehat{y_{p_k}}, ..., y_{p_3}) = 0;$$

$1 \leq i \leq m$*;* $1 \leq p_0 < p_1 < p_2 < p_3 \leq m$*. Here,* $\hat{x}$ *means* x *is removed from the expression.*

For SL_2-matrices, we get:

COROLLARY 17. *Consider* $\mathbb{C}[SL_2 \times ... \times SL_2]^{SL_2}$*, where* SL_2 *acts by simultaneous conjugation on* m *matrices. Then*

(i) this algebra of invariants is generated by $\mathit{Tr}x_j$*,* $1 \leq j \leq m$*,* $\mathit{Tr}x_j x_k$*,*$1 \leq j < k \leq m$*, and*

$$s_3(x_{j_1}, x_{j_2}, x_{j_3}) := x_{j_1}, x_{j_2}, x_{j_3} - x_{j_1}, x_{j_3}, x_{j_2};\ 1 \leq j_1 < j_2 < j_3 \leq m;$$

(ii) the defining relations for the invariant ring are

$$\mathit{Tr}s_3(x_{i_1}x_{i_2}x_{i_3}) \cdot \mathit{Tr}s_3(x_{j_1}x_{j_2}x_{j_3}) + \\ 2\begin{vmatrix} \mathit{Tr}x_{i_1}x_{j_1} - \frac{1}{2}\mathit{Tr}x_{i_1}\,\mathit{Tr}x_{j_1} & \cdots & \mathit{Tr}x_{i_1}x_{j_3} - \frac{1}{2}\mathit{Tr}x_{i_1}\,\mathit{Tr}x_{j_3} \\ \vdots & \ddots & \vdots \\ \mathit{Tr}x_{i_3}x_{j_1} - \frac{1}{2}\mathit{Tr}x_{i_3}\,\mathit{Tr}x_{j_1} & \cdots & \mathit{Tr}x_{i_3}x_{j_3} - \frac{1}{2}\mathit{Tr}x_{i_3}\,\mathit{Tr}x_{j_3} \end{vmatrix} = 0;$$

$1 \leq i_1 < i_2 < i_3 \leq m$*;* $1 \leq j_1 < j_2 < j_3 \leq m$*; and*

$$\sum_{k=0}^{3}(-1)^k \left(\mathit{Tr}x_i x_{p_k} - \frac{1}{2}\mathit{Tr}x_i\,\mathit{Tr}x_{p_k}\right)\mathit{Tr}s_3(y_{p_0}, ..., \widehat{y_{p_k}}, ..., y_{p_3}) = 0;$$

$1 \leq i \leq m$*;* $1 \leq p_0 < p_1 < p_2 < p_3 \leq m$*; here,* $\mathit{Tr}x_i^2$ *should be read as* $2 \cdot \big((\mathit{Tr}x_i)^2 - 1\big)$*, and* $\hat{x}$ *means* x *is removed from the expression.*

PROOF. Proposition 6 tells us that, letting $(I') = (\det y_1 + \frac{1}{4}t_1^2 - 1, ..., \det y_m + \frac{1}{4}t_m^2 - 1)$, we have:

$$\mathbb{C}[SL_2^m]^{SL_2} \cong \left(\mathbb{C}[M_0^m]^{SL_2} \otimes \mathbb{C}[t_1, ..., t_m]\right)/(I').$$

Note that in this right coordinate ring, we have

$$\mathrm{Tr}y_i^2 = -2\det y_i = \frac{1}{2}t_i^2 - 2.$$

We can now eliminate the generators $\mathrm{Tr}y_i^2$ for $\mathbb{C}[M_0^m]^{SL_2}$ with the ideal (I'), and use the isomorphism (1.1.3) and Lemma 4 to translate the results to $\mathbb{C}[SL_2^m]^{SL_2}$. Doing this gives the Corollary as stated. □

For the rest of this text we will often consider the case of 3 matrices to calculate with. This is how the Corollary looks like in this case:

COROLLARY 18. *Consider* $\mathbb{C}[SL_2 \times SL_2 \times SL_2]^{SL_2}$*, where* SL_2 *acts by simultaneous conjugation on* 3 *matrices. Then:*

(i) this algebra of invariants is generated by Trx_1, Trx_2, Trx_3, Trx_1x_2, Trx_1x_3, Trx_2x_3, $Tr(x_1x_2x_3 - x_1x_3x_2)$*;*

(ii) the only defining relation of the ring with respect to the above generator is:

$$\begin{aligned} 0 \quad = \quad & \left(Tr(x_1x_2x_3 - x_1x_3x_2)\right)^2 + \\ & 2\begin{vmatrix} \frac{1}{2}(Trx_1)^2 - 2 & Trx_1x_2 - \frac{1}{2}Trx_1Trx_2 & Trx_1x_3 - \frac{1}{2}Trx_1Trx_3 \\ Trx_1x_2 - \frac{1}{2}Trx_1Trx_2 & \frac{1}{2}(Trx_2)^2 - 2 & Trx_2x_3 - \frac{1}{2}Trx_2Trx_3 \\ Trx_1x_3 - \frac{1}{2}Trx_1Trx_3 & Trx_2x_3 - \frac{1}{2}Trx_2Trx_3 & \frac{1}{2}(Trx_3)^2 - 2 \end{vmatrix}. \end{aligned}$$

More concretely, letting $a = Trx_1$*,* $b = Trx_2$*,* $c = Trx_3$*,* $d = Trx_1x_2$*,* $e = Trx_1x_3$*,* $f = Trx_2x_3$*,* $g = Trx_1x_2x_3 - Trx_1x_3x_2$*, the algebra of invariants can be seen as the following algebraic structure:*

$$\mathbb{C}[a, b, c, d, e, f, g]/(rel),$$

where

$$\begin{aligned} rel \quad = \quad & g^2 + 4a^2 + 4b^2 + 4c^2 + 4d^2 + 4e^2 + 4f^2 \\ & -4ace - 4abd - 4bcf + 4def \\ & -b^2e^2 - a^2f^2 - c^2d^2 \\ & +2a^2bcf + 2abc^2d + 2ab^2ce \\ & -2bcde - 2acdf - 2abef \\ & -a^2b^2c^2 - 16. \end{aligned} \tag{1.3.3}$$

CHAPTER 2

A projective quotient

Instead of looking at the algebra of invariants of the the affine varieties $\mathbb{C}[SL_2^m]$, we can also embed this space into some projective variety. This enables us to interpret the concept of a "quotient" geometrically.

FIrst, in Section 2.1, we define such an an embedding of $\mathbb{C}[SL_2^m]$ into a projective space. Next, in Section 2.2, we describe some general definitions and theory for the construction of quotients on (projective) varieties. In the rest of the chapter, we construct a cover for all non-nilpotent matrices in our projective space. For this cover, we can calculate the invariants and consequently obtain a so-called "good quotient".

2.1. An embedding

In this chapter, we will focus on one particular embedding. Letting

$$Q = \{(a : b : c : d : \Delta) \subset \mathbb{P}^4 \mid ad - bc = \Delta^2\},$$

our embedding is:

$$\begin{array}{rcl} SL_2^m & \overset{\phi}{\hookrightarrow} & Q \times \ldots \times Q \\ \left(\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} a_m & b_m \\ c_m & d_m \end{smallmatrix}\right)\right) & \mapsto & \left((a_1 : b_1 : c_1 d_1 : 1), \ldots, (a_m : b_m : c_m : d_m : 1)\right). \end{array}$$

Thus, in effect we add the square roots of the determinants of the SL_2 matrices as extra coordinates to obtain something projective. The SL_2-action on SL_2^m then naturally extends to $Q \times \ldots \times Q$ by simultaneously conjugating the matrices formed by a_i, b_i, c_i, d_i and leaving the Δ_i intact. $Q \times \ldots \times Q$ is naturally a projective variety via the embedding in $\mathbb{P}^{5^m-1}$ with basis consisting of m-fold products of $a_i, b_i, c_i, d_i, \Delta_i$'s for $i = 1, \ldots, k$; for example, $Q \times Q$ embeds into $\mathbb{P}^{24}$:

$$\begin{array}{rcl} Q \times Q & \hookrightarrow & \mathbb{P}^{24} \\ ((a_1 : b_1 : c_1 : d_1 : \Delta_1), (a_2 : b_2 : c_2 : d_2 : \Delta_2)) & \mapsto & (a_1 a_2 : a_1 b_2 : a_1 c_2 : a_1 d_2 : \\ & & a_1 \Delta_2 : \ldots : \Delta_1 a_2 : \Delta_1 b_2 : \\ & & \Delta_1 c_2 : \Delta_1 d_2 : \Delta_1 \Delta_2). \end{array}$$

Our SL_2-action then naturally extends to a SL_2-action on $\mathbb{P}^{5^m-1}$.

The embedding of $X = SL_2^m$ is a so-called *compactification*: an embedding X of as a dense subset of a compact space.

Note that our choice of embedding is purely heuristic: there are many other possible projective spaces in which SL_2^m could be embedded. We chose this one because it seems intuitive: it preserves the separate matrices and the new coordinates we add have a concrete interpretation as the square root of the determinant. Another obvious choice would be to embed SL_2^m into a $\mathbb{P}^{4m}$ with a map:

$$(a_1, b_1, c_1, d_1, \ldots, d_m) \mapsto (a_1 : b_1 : \ldots : d_m : 1).$$

We will not study alternative embeddings in this text.

2.2. Quotients

We now use geometric invariant theory (GIT) to geometrically study the orbits of the SL_2-action. The ultimate goal in geometric invariant theory (GIT) is to construct a so-called *geometric quotient.* Geometric invariant theory was developed by David Mumford in 1965, and his book [**10**] is the primary source about this field. The definitions we give below were taken mainly from [**8**] and [**9**, Ch.6]: two sources where the concepts of GIT are introduced in a more accessible way.

Definition 19. Let X be an algebraic variety with an action by a reductive group G. A pair (Y, ϕ) is called a *categorical quotient* if:
(i) Y is an algebraic variety;
(ii) ϕ is a morphism of varieties from X to Y;
(iii) ϕ is G-invariant;
(iv) ϕ has the *universal mapping property*: given any algebraic variety Z and a morphism $\psi : X \to Z$ which is constant on G-orbits, there exists a unique $\theta : Y \to Z$ such that $\psi = \theta \circ \phi$; that is, we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[dr, "\psi"'] & Y \arrow[d, "\theta"] \\
 & Z,
\end{tikzcd}$$

The important property is of course (iv): it indicates that the map separates the orbits of X under G as much as possible. If such a categorical quotient (Y, ϕ) exists, we denote it by $X//G$. Also, we note that up to isomorphism, categorical quotients must be unique.

We can also define a *good quotient*, which satisfies some stronger conditions:

Definition 20. Let X be an algebraic variety with an action by a reductive group G. A pair (Y, ϕ) consisting of an algebraic variety Y and a morphism ϕ is called a *good quotient* if it satisfies the following conditions:
(i) ϕ is surjective;
(ii) ϕ is G-invariant;
(iii) ϕ is an affine morphism (ie, the inverse image of an open affine set is again an open affine set);
(iv) if $W \subset X$ is G-stable and closed, then $\phi(W)$ is closed;
(v) let $X_1, X_2 \subset X$ disjoint, G-stable and closed, then $\phi(W_1) \cap \phi(W_2) = \emptyset$;
(vi) for open $U \subset Y$, $\phi^* : k[U] \to k[\phi^{-1}(U)]^G$ is an isomorphism.

One can show that a good quotient (Y, ϕ) is also a categorical quotient. Note that even if (Y, ϕ) might separate the orbits as much as possible, it might not separate them perfectly. If it does, it is called a geometric quotient:

Definition 21. Let (Y, ϕ) be a good quotient. If the set of points of Y is in bijection with the set of orbits in X, then (Y, ϕ) is called a *geometric quotient.* This latter statement is equivalent to all orbits of X under G being closed.

2.2.1. Good quotients: the affine case. Now if X is an affine variety and G is a so-called "reductive group" (SL_2 is such a reductive group), then we know what the categorical quotient looks like. For this, we need the construction from algebraic geometry of the *spectrum* of a ring, denoted Spec$R$. This is a construction to create an affine variety having R as its coordinate ring.

So, if we have an algebra of invariants $k[W]^{SL_2}$, then Spec$k[W]^{SL_2}$ is an affine variety with coordinate ring $k[W]^{SL_2} \subset k[W]$, and this inclusion induces a map

$\phi : W \to \mathrm{Spec}k[W]^{SL_2}$: given a point it simply gives the values of the invariant functions evaluated in that point. One can then show that ϕ is a good quotient for SL_2 acting on W. For example, if we take $W = SL_2 \times SL_2$, we know $\mathbb{C}[SL_2^2]^{SL_2} = \mathbb{C}[\mathrm{Tr}x_1, \mathrm{Tr}x_2, \mathrm{Tr}x_1x_2]$. So a good quotient for $SL_2 \times SL_2$ is $(\mathbb{C}^3, \phi)$, where $\phi(x_1, x_2) = (\mathrm{Tr}x_1, \mathrm{Tr}x_2, \mathrm{Tr}x_1x_2)$.

2.2.2. Good quotients: the projective case. It turns out that we can not, in general, construct a good quotient for a complete projective variety X: the best we can do is to construct a quotient on some open subvariety of X. For this, we can use the good quotients we are able to construct for affine varieties by constructing a cover, as explained below.

In topology, a *cover* of a set X is an indexed family of sets

$$U_A = \{U_\alpha \,|\, \alpha \in A\}$$

such that

$$X \subseteq \bigcup_{\alpha \in A} U_\alpha;$$

an *open cover* is a cover of which all U_α are open (see [**14**]). For example, in our embedding, $\phi(SL_2^m)$ is obviously an open subset of $Q \times ... \times Q$; it is also affine, dense and SL_2-stable.

Now suppose we have a cover by such open, affine, SL_2-stable subsets. Then for each of the subsets, we can calculate its coordinate ring $\mathbb{C}[U_\alpha]$ and algebra of invariants $\mathbb{C}[U_\alpha]^{SL_2}$. As indicated above, this induces good quotients $U_\alpha \to \mathrm{Spec}\mathbb{C}[U_\alpha]^{SL_2}$. One can then check that the map $\phi : X' \to \mathrm{Spec}X'^G$ obtained by combining all ϕ_α is a good quotient for X'.

2.3. The case $m = 1$

In the case $m = 1$, our embedding looks like this. Letting $A_0 = SL_2$, we have:

$$\begin{array}{ccl} A_0 & \overset{\phi}{\hookrightarrow} & Q = \{(a : b : c : d : \Delta) \subset \mathbb{P}^4 \,|\, ad - bc = \Delta^2\}, \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) & \mapsto & (a : b : c : d : 1)\,. \end{array}$$

As mentioned,

$$U_0 = \phi(A_0) = \{(a : b : c : d : \Delta) \in Q \,|\, \Delta \neq 0\} \tag{2.3.1}$$

is an open, affine, dense, SL_2-stable subset of Q, and since this is the case that we have been studying all along, we know what happens here: the coordinate ring of A_0 is

$$\mathbb{C}[a', b', c', d']/(a'd' - b'c' = 1),$$

and letting $y = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$, its ring of invariants is simply $\mathbb{C}[\mathrm{Tr}y]$. The induced good quotient is simply $(\mathbb{C}, \phi)$, where $\phi(M) = \mathrm{Tr}M$. We can also see this as a good quotient on a part of Q: for as a function on U_0, the fuction $\mathrm{Tr}y$ is $\frac{a+d}{\Delta}$. So equivalently, letting $Y = \{(a : b) \in \mathbb{P}^1 \,|\, b \neq 0\}$, $\phi(a : b : c : d : \Delta) = (a + d : \Delta)$, we have that (Y, ϕ) is a good quotient for U_0.

We obtain another obvious subset of Q with the same properties by looking at

$$U_1 = \{(a'' : b'' : c'' : d'' : \Delta'') \in Q \,|\, a'' + d'' \neq 0\}. \tag{2.3.2}$$

To get a coordinate ring for A_1, we can assume $a'' + d'' = 1$ to obtain the embedding of the affine set

$$A_1 = \left\{ \left(\begin{pmatrix} a'' & b'' \\ c'' & 1 - a'' \end{pmatrix}, z'' \right) \,|\, z''^2 = a'' - a''^2 - b''c'' \right\}$$

into Q. This space has coordinate ring $\mathbb{C}[a'', b'', c'', z'']/(a'' - a''^2 - b''c'' = z''^2)$, and SL_2 acts by conjugating the matrix and leaving the z'' intact. To obtain the ring of invariants, we proceed in the usual way by constructing an isomorphism of SL_2-representations to $M_0 \oplus \mathbb{C}$. One checks that this is done by

$$\begin{array}{ccc} A_1 & \overset{\psi}{\to} & M_0 \oplus \mathbb{C} \\ \left(\begin{pmatrix} a'' & b'' \\ c'' & 1-a'' \end{pmatrix}, z'' \right) & \mapsto & \left(\begin{pmatrix} a'' - \frac{1}{2} & b'' \\ c'' & \frac{1}{2} - a'' \end{pmatrix}, z'' \right), \end{array}$$

and that the coordinate ring of $\psi(A_1)$ is

$$\mathbb{C}[a,b,c,z]/(-a^2 - bc + \frac{1}{4} = z^2) \cong \mathbb{C}[a,b,c] \otimes (1 \oplus z).$$

The right-hand side has ring of invariants $\mathbb{C}[\mathrm{Tr} x^2] \otimes (1 \oplus z)$, translating to $\mathbb{C}[z]$ for the left-hand side. This gives algebra of invariants $\mathbb{C}[A_1]^{SL_2} = \mathbb{C}[z'']$.

Now note that as a function on U_1, z'' corresponds to $\frac{\Delta}{a''+d''}$. Similarly to above, a good quotient for U_1 is then $Y = \{(a:b) \in \mathbb{P}^1 \,|\, a \neq 0\}$, $\phi(a:b:c:d:\Delta) = (a+d:\Delta)$.

2.3.1. Constructing a quotient. Now it is clear how a good quotient for $U_0 \cup U_1$ should look like: we claim this is simply $Y = \mathbb{P}^1$, $\phi(a:b:c:d:\Delta) = (a+d:\Delta)$. For we note that restricted to U_0 and U_1 we get back our good quotients for U_0, U_1. This is enough to prove the claim.

Letting $A = \mathbb{B}$, we now have a cover U_A. We note that the points that in this cover are exactly those points for which either $a+d \neq 0$ or $\Delta \neq 0$: the non-nilpotent matrices. Denote this set by Q^{nn}.

Thus, up to conjugation, the only point in Q that is not in this cover is $n = (0:1:0:0:0)$ to a nilpotent matrix. Now consider the embedding of $\mathbb{P}^4$ in $\mathbb{A}^5$ and consider the set of points

$$(0, \alpha, 0, 0, 0); \;\; \alpha \neq 0$$

corresponding to n: the "lifts" of n. Suppose they belong to some closed set $V(S) \subset \mathbb{A}^5$, then we note that also $(0,0,0,0,0) \in V(S)$: for if we restrict any polynomial defining the closed set to

$$S = \{(0, \alpha, 0, 0, 0) \,|\, \alpha \in \mathbb{C}\}$$

then it can have only finitely many zeros, so in fact the polynomial defining the closed set must be constant restricted to S and also include $\alpha = 0$. We will see later that in geometric invariant theory, it is exactly these points with 0 in the closure of the lifts that will be ignored in the general construction of a good quotient.

We summarize our results on covering Q in the following proposition:

Proposition 22. *Letting $A = \mathbb{B}$, we have a cover U_A of*

$$Q^{nn} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \Delta \in Q \,|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is not nilpotent} \right\}$$

given by (2.3.1) and (2.3.2). We get a good quotient $Q^{nn} \to \mathbb{P}^2$ given by $\phi(a:b:c:d:\Delta) = (a+d:\Delta)$.

2.3.2. Glueing the algebras of invariants. Another way of looking at our problem is as follows. We have two affine spaces A_0 and A_1 embedded into Q. Thus, we can try to construct a coordinate ring for $\mathbb{C}[A_0 \cup A_1]$. This is done simply by tracking the embeddings of A_0 and A_1 into Q, adding coordinates to the coordinate rings as needed. We obtain the following two isomorphic coordinate rings:

$$\begin{aligned}
\mathbb{C}[a', b', c', d', \frac{1}{a'+d'}] &\cong \mathbb{C}[a'', b'', c'', z'', \frac{1}{z''}] \\
a', b', c', d' &\mapsto \frac{a''}{z''}, \frac{b''}{z''}, \frac{c''}{z''}, \frac{1-a''}{z''} \\
\frac{1}{a'+d'} &\mapsto z'' \\
a'', b'', c'' &\mapsto \frac{a'}{a'+d'}, \frac{b'}{a'+d'}, \frac{c'}{a'+d'} \\
\tilde{z_1} &\mapsto \frac{1}{a'+d'}.
\end{aligned}$$

But this also induces an isomorphism between the coordinate rings:

$$\begin{aligned}
\mathbb{C}[\mathrm{Tr}y, \frac{1}{\mathrm{Tr}y}] &\cong \mathbb{C}[z'', \frac{1}{z''}] \\
\mathrm{Tr}y &\leftrightarrow \frac{1}{z''}.
\end{aligned}$$

In this case, since the functions $\mathrm{Tr}y$ and $\frac{1}{\mathrm{Tr}y}$ as a function on Q looks like

$$\frac{a+d}{\Delta}, \frac{\Delta}{a+d}$$

and similarly for z'' and $\frac{1}{z''}$, we see that this approach also motivates the good quotient $(M : \Delta) \mapsto (\mathrm{Tr}M : \Delta)$. However, note that we do need the cover as described above to be able to show that this map does actually satisfy the required properties.

2.4. A cover for $(Q \times ... \times Q)^{nn}$

We can generalize the ideas used in the $m = 1$ case to construct a cover for $Q \times ... \times Q$ in general. Now, we can choose for each of the m matrices in the tuple independently whether we choose the trace or determinant to be non-zero.

Let us formalize this. Let $A = \mathbb{B}^m$, $f_{i,0} = \Delta_i$, $f_{i,1} = a_i + d_i$. Then our cover is

$$\bigcup_{a \in \mathbb{B}^m} U_a,$$

where

$$\begin{aligned}
\text{(2.4.1)} \quad U_{(a_1,...,a_n)} &= \{(a_1 : b_1 : c_1 : d_1 : \Delta_1), ..., (a_m : b_m : c_m : d_m : \Delta_m) \\
&\quad \in Q \times Q \times Q \mid f_{1,a_1} \cdot f_{2,a_2} \cdot ... \cdot f_{m,a_m} \neq 0\}.
\end{aligned}$$

Clearly, this covers exactly all the points in $Q \times ... \times Q$ corresponding to tuples of matrix, none of which are nilpotent.

Now it is also clear what affine space an U_a corresponds to. Let M_1 be the space of trace 1 matrices along with their determinant:

$$M_1 = \left\{ \left(\left(\begin{array}{cc} a & b \\ c & 1-a \end{array} \right), d \right) \mid d^2 = a - a^2 - bc \right\},$$

where SL_2 acts on M_1 by conjugating the matrix and leaving the d intact. Then each $\Delta_i \neq 0$ gives an SL_2 using the usual embedding $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (a : b : c : d : 1)$; each $a_i + d_i \neq 0$ gives a M_1 using the embedding $\left(\left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right), d\right) \mapsto (a : b : c : 1-a : d)$. Thus, letting $U_{a,i} = SL_2$ if $a_i = 0$; $U_{a,i} = M_1$ if $a_i = 1$, we see $U_a \cong U_{a,1} \times ... \times U_{a,m}$.

It follows that $\mathbb{C}[U_a] = \mathbb{C}[a_1, b_1, c_1, d_1, ..., c_m, d_m]/(f_1, f_2, ..., f_m)$, where

$$f_i = \begin{cases} a_i d_i - b_i c_i - 1 & \text{if } a_i = 0; \\ d_i^2 - a_i + a_i^2 + b_i c_i & \text{if } a_i = 1. \end{cases}$$

By our usual arguments, for the algebra of invariants we then have $\mathbb{C}[U_a]^{SL_2} = (\mathbb{C}[M_o^m]^{SL_2} \otimes \mathbb{C}[t_1, ..., t_m])/(g_1, ..., g_m)$, where

$$g_i = \begin{cases} \det y_1 + \frac{1}{4}t_1^2 - 1 & \text{if } a_i = 0; \\ -a_i^2 - b_i c_i + \frac{1}{4} - t_i^2 & \text{if } a_i = 1. \end{cases}$$

Letting x_i be the obvious generic matrices for SL_2 and M_1, it is an easy verification that translating back our results is as usual:

LEMMA 23. *Let ϕ be the isomorphism*

$$(\mathbb{C}[M_o^m]^{SL_2} \otimes \mathbb{C}[t_1, ..., t_m])/(g_1, ..., g_m) \to \mathbb{C}[U_a]^{SL_2}$$

as above. Suppose $a_i = a_j = 1$, $a_k = a_l = 0$. Then

$$\begin{aligned} \phi(t_i) &= d_i \\ \phi(t_k) &= \mathrm{Tr} x_k \\ \phi(\mathrm{Tr} y_i y_j) &= \mathrm{Tr} x_i x_j - \frac{1}{2}; \\ \phi(\mathrm{Tr} y_i y_k) &= \mathrm{Tr} x_i x_k - \frac{1}{2}\mathrm{Tr} x_k; \\ \phi(\mathrm{Tr} y_k y_l) &= \mathrm{Tr} x_k x_l - \frac{1}{2}\mathrm{Tr} x_k \mathrm{Tr} x_l; \\ \phi(\mathrm{Tr}(y_a y_b y_c - y_a y_c y_b) &= \mathrm{Tr}(x_a x_b x_c - x_a x_c x_b). \end{aligned}$$

Also, in $\mathbb{C}[U_a]^{SL_2}$ the following holds:

$$\begin{aligned} \mathrm{Tr} x_i^2 &= 1 - d_i^2 \\ \mathrm{Tr} x_k^2 &= (\mathrm{Tr} x_k)^2 - 2, \end{aligned}$$

so the elimination of the $\mathrm{Tr} x_i^2$ generators is always possible.

2.5. The case $m = 2$

Let us take the cover by $\mathbb{B}^2$ of $(Q \times Q)^{nn}$ as defined above. For example, we get an affine open set

$$U_{(0,0)} = \{((M_1 : \Delta_1), (M_2 : \Delta_2)) \mid \Delta_1 \Delta_2 \neq 0\}$$

which can be identified with SL_2^2 with coordinate ring $\mathbb{C}[M_1, M_2]$ and algebra of invariants $\mathbb{C}[\mathrm{Tr} M_1, \mathrm{Tr} M_2, \mathrm{Tr} M_1 M_2]$. As functions on $Q \times Q$ these generators are

$$\frac{\mathrm{Tr} M_1 \cdot \Delta_2}{\Delta_1 \cdot \Delta_2}, \frac{\Delta_1 \cdot \mathrm{Tr} M_2}{\Delta_1 \cdot \Delta_2}, \frac{\mathrm{Tr} M_1 M_2}{\Delta_1 \cdot \Delta_2},$$

so we have a good quotient $U_{(0,0)} \to \{(a : b : c : d) \in \mathbb{P}^3 \mid d \neq 0\}$:

$$((M_1 : \Delta_1), (M_2 : \Delta_2)) \mapsto (\mathrm{Tr} M_1 \Delta_2 : \Delta_1 \mathrm{Tr} M_2 : \mathrm{Tr} M_1 M_2 : \Delta_1 \Delta_2).$$

Similarly, for $U_{(1,0)}$ with coordinate ring $\mathbb{C}[y_1, d_1, y_2]$, with y_1 a trace 1 matrix and y_2 an SL_2 matrix, we can apply Lemma 23 to get $\mathbb{C}[y_1, d_1, y_2]^{SL_2} = \mathbb{C}[d_1, \mathrm{Tr} y_2, \mathrm{Tr} y_1 y_2]$. On $Q \times Q$ these generators correspond to, respectively:

$$\frac{\Delta_1 \Delta_2}{\mathrm{Tr} M_1 \Delta_2}, \frac{\mathrm{Tr} M_1 \mathrm{Tr} M_2}{\mathrm{Tr} M_1 \Delta_2}, \frac{\mathrm{Tr} M_1 M_2}{\mathrm{Tr} M_1 \Delta_2}.$$

This gives a good quotient to $\{(a : b : c : d) \in \mathbb{P}^3 \mid d \neq 0\}$ given by:

$$((M_1 : \Delta_1), (M_2 : \Delta_2)) \mapsto (\mathrm{Tr} M_1 \mathrm{Tr} M_2 : \mathrm{Tr} M_1 M_2 : \Delta_1 \Delta_2 : \mathrm{Tr} M_1 \Delta_2).$$

Similarly, for the other affine parts $U_{(0,1)}$ and $U_{(1,1)}$ the resulting good quotient to $\{(a : b : c : d) \in \mathbb{P}^3 \,|\, d \neq 0\}$ is as follows:

$$\begin{array}{rcl} ((M_1 : \Delta_1), (M_2 : \Delta_2)) & \mapsto & (\mathrm{Tr}M_1\mathrm{Tr}M_2 : \mathrm{Tr}M_1M_2 : \Delta_1\Delta_2 : \Delta_1\mathrm{Tr}M_2); \\ ((M_1 : \Delta_1), (M_2 : \Delta_2)) & \mapsto & (\Delta_1\mathrm{Tr}M_2 : \mathrm{Tr}M_1\Delta_2 : \mathrm{Tr}M_1M_2 : \mathrm{Tr}M_1\mathrm{Tr}M_2). \end{array}$$

But then let us consider the map

$$\begin{array}{rcl} (2.5.1)\; \phi : (Q \times Q)^{nn} & \rightarrow & \phi((Q \times Q)^{nn}) \subset \mathbb{P}^4 \\ (M_1 : \Delta_1, M_2 : \Delta_2) & \mapsto & (\mathrm{Tr}M_1\Delta_2 : \Delta_1\mathrm{Tr}M_2 : \mathrm{Tr}M_1\mathrm{Tr}M_2 : \Delta_1\Delta_2 : \mathrm{Tr}M_1M_2). \end{array}$$

Restricting this map to $U_{(0,0)}$, for instance, gives:

$$\phi : U_{(0,0)} \rightarrow \{(a : b : c : d : e) \in \mathbb{P}^4 \,|\, ab = cd \wedge d \neq 0\}.$$

But now note that via $\psi : (a : b : c : d : e) \mapsto (a : b : e : d)$ we have a commutative diagram

$$\begin{tikzcd} U_{(0,0)} \arrow[r] \arrow[d, "\phi"'] & \{(a : b : c : d) \in \mathbb{P}^3 \,|\, d \neq 0\} \\ \phi(U_{(0,0)}) \arrow[ur, dashed, "\cong", "\psi"'] & \end{tikzcd}$$

so ϕ is a good quotient for $U_{(0,0)}$. Similar conclusions can be drawn for the other affine parts. Finally, we arrive at the conclusion:

Proposition 24. Let

$$Y = \left\{(x_1 : x_2 : x_3 : x_4 : x_5) \in \mathbb{P}^4 \,|\, x_1x_2 = x_3x_4\right\} \setminus (0 : 0 : 0 : 0 : 1),$$

ϕ as above. Then (Y, ϕ) is a good quotient for $(Q \times Q)^{nn}$.

Note that some parts of $Q \times Q$ are now not covered for which an affine subset could be defined: for example, consider

$$\{((M_1 : \Delta_1), (M_2 : \Delta_2)) \,|\, \mathrm{Tr}M_1M_2 \neq 0\}.$$

However, in this case it is not obvious how to calculate the invariants as the normal method with representation isomorphisms doesn't seem to work. We will come back to the question of making a cover as large as possible in the next Chapter.

Another remark we make is that our good quotient is definitely not a geometric quotient: for example, note that

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : 1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : 1\right) \mapsto (2 : 2 : 4 : 1 : 2)$$

but also

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : 1, \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} : 1\right) \mapsto (2 : 2 : 4 : 1 : 2)$$

and these points are certainly not in the same orbit.

2.6. The case $m = 3$

In this case, we have an embedding

$$\begin{array}{rcl} SL_2^3 & \overset{\phi}{\hookrightarrow} & Q \times Q \times Q \\ \left(\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_3 & b_3 \\ c_3 & d_3 \end{smallmatrix}\right)\right) & \mapsto & ((a_1 : b_1 : c_1d_1 : 1), (a_2 : b_2 : c_2 : d_2 : 1), \\ & & ((a_3 : b_3 : c_3 : d_3 : 1))\,. \end{array}$$

2.6.1. Constructing a good quotient. Similarly to the $Q \times Q$ case, we will look at our affine covering indexed by $\mathbb{B}^3$ and see what maps the invariant functions induce. We did not perform these calculations in detail, we will be a bit brief.

As an example, for the embedding $U_{(0,0,0)}$ of $SL_2 \times SL_2 \times SL_2$, we get an induced map on

$$\{(M_1 : \Delta_1, M_2 : \Delta_2, M_3 : \Delta_3) \in Q \times Q \times Q \,|\, \Delta_1\Delta_2\Delta_3 \neq 0\}$$

defined by:

$$\begin{aligned}((M_1 : \Delta_1), (M_2 : \Delta_2), (M_3 : \Delta_3)) \quad \mapsto \quad & (\mathrm{Tr}M_1\Delta_2\Delta_3 : \Delta_1\mathrm{Tr}M_2\Delta_3 : \Delta_1\Delta_2\mathrm{Tr}M_3 : \\ & \mathrm{Tr}M_1M_2\Delta_3 : \Delta_2\mathrm{Tr}M_1M_3 : \Delta_1\mathrm{Tr}M_2M_3 : \\ & \mathrm{Tr}(M_1M_2M_3 - M_1M_3M_2) : \Delta_1\Delta_2\Delta_3).\end{aligned}$$

Its image is

$$\{(a : b : c : d : e : f : g : h) \in \mathbb{P}^7 \,|\, h \neq 0 \wedge f = 0\},$$

where f is a projective equivalent of the normal three-matrix relation (1.3.3).

As another example, consider the embedding $U_{(1,1,1)}$ of $M_1 \times M_1 \times M_1$. One verifies that this gives an induced map on

$$\{(M_1 : \Delta_1, M_2 : \Delta_2, M_3 : \Delta_3) \in Q \times Q \times Q \,|\, \mathrm{Tr}M_1 \cdot \mathrm{Tr}M_2 \cdot \mathrm{Tr}M_3 \neq 0\}$$

defined by

$$\begin{aligned}((M_1 : \Delta_1), (M_2 : \Delta_2), (M_3 : \Delta_3)) \quad \mapsto \quad & (\Delta_1\mathrm{Tr}M_2\mathrm{Tr}M_3 : \mathrm{Tr}M_1\Delta_2\mathrm{Tr}M_3 : \\ & \mathrm{Tr}M_1\mathrm{Tr}M_2\Delta_3 : \mathrm{Tr}M_1M_2\mathrm{Tr}M_3 : \\ & \mathrm{Tr}M_1M_3\mathrm{Tr}M_2 : \mathrm{Tr}M_2M_3\mathrm{Tr}M_3 : \\ & \mathrm{Tr}(M_1M_2M_3 - M_1M_3M_2) : \\ & \mathrm{Tr}M_1\mathrm{Tr}M_2\mathrm{Tr}M_3).\end{aligned}$$

Again, the image is

$$\{(a : b : c : d : e : f : g : h) \in \mathbb{P}^7 \,|\, h \neq 0 \wedge f = 0\},$$

for some suitable function f relating g to the other invariants.

Indeed, we see that these two good quotients already provide all the "obvious" polynomials linear in each of the three matrices: the eight combinations

$$i_1, ..., i_8 = \{\mathrm{Tr}M_1, \Delta_1\} \cdot \{\mathrm{Tr}M_2, \Delta_2\} \cdot \{\mathrm{Tr}M_3, \Delta_3\},$$

the combinations with traces of two matrices

$$i_9, i_{10} = \mathrm{Tr}M_1M_2 \cdot \{\mathrm{Tr}M_3, \Delta_3\};\ i_{11}, i_{12} = \mathrm{Tr}M_1M_3 \cdot \{\mathrm{Tr}M_2, \Delta_2\};$$
$$i_{13,14} = \mathrm{Tr}M_2M_3 \cdot \{\mathrm{Tr}M_1, \Delta_1\},$$

and finally $i_{15} = (\mathrm{Tr}M_1M_2M_3 - M_1M_3M_2)$. The other affine parts give the same polynomials, but with a different one of the $\{i_1, ..., i_8\}$ polynomials being non-zero on the image.

In all, this gives us a function ϕ:$(Q \times Q \times Q)^{nn} \to \mathbb{P}^{14}$ which we claim is a good quotient to $\phi((Q \times Q \times Q)^{nn})$:

$$\begin{aligned}(M_1 : \Delta_1, ..., M_3 : \Delta_3) \quad \overset{\phi}{\mapsto} \quad & (\mathrm{Tr}M_1\mathrm{Tr}M_2\mathrm{Tr}M_3 : \mathrm{Tr}M_1\mathrm{Tr}M_2\Delta_3 : \mathrm{Tr}M_1\Delta_2\mathrm{Tr}M_3 : \\ & \mathrm{Tr}M_1\Delta_2\Delta_3 : \Delta_1\mathrm{Tr}M_2\mathrm{Tr}M_3 : \Delta_1\mathrm{Tr}M_2\Delta_3 : \\ & \Delta_1\Delta_2\mathrm{Tr}M_3 : \Delta_1\Delta_2\Delta_3 : \\ & \mathrm{Tr}M_1M_2\mathrm{Tr}M_3 : \mathrm{Tr}M_1M_2\Delta_3 : \\ & \mathrm{Tr}M_1M_3\mathrm{Tr}M_2 : \mathrm{Tr}M_1M_3\Delta_2 : \\ & \mathrm{Tr}M_2M_3\mathrm{Tr}M_1 : \mathrm{Tr}M_2M_3\Delta_1 : \\ & \mathrm{Tr}(M_1M_2M_3 - M_1M_3M_2)).\end{aligned}$$

So what is the image of ϕ? First, note that obviously there are some relations holding between the invariants used in the map. If we list them, then we see that $\phi((Q \times Q \times Q)^{nn}) \subset Y'$, where

$$\begin{aligned} Y' \quad = \quad & \{(i_1 : i_2 : ... : i_{15}) \in \mathbb{P}^{14} \,| \\ & i_1 i_4 = i_2 i_3, i_1 i_6 = i_2 i_5, i_1 i_7 = i_3 i_5, i_2 i_8 = i_4 i_6, i_3 i_8 = i_4 i_7, i_5 i_8 = i_6 i_7, \\ & i_1 i_8 = i_2 i_7 = i_3 i_6 = i_4 i_5, \\ & i_9 i_1 = i_{10} i_2, i_9 i_3 = i_{10} i_4, i_9 i_5 = i_{10} i_6, i_9 i_7 = i_{10} i_8, \\ & i_{11} i_1 = i_{12} i_3, i_{11} i_{11} i_2 = i_{12} i_4, i_{11} i_5 = i_{12} i_7, i_{11} i_6 = i_{12} i_8, \\ & i_{13} i_1 = i_{14} i_5, i_{13} i_2 = i_{14} i_6, i_{13} i_3 = i_{14} i_7, i_{13} i_4 = i_{14} i_8, \\ & i_8^2 i_{15}^2 = i_4^2 i_{12}^2 + ... + 4 i_8 i_{10} i_{14}\}. \end{aligned}$$

Also, note that each of the affine parts is defined by demanding that one of the $i_1, ..., i_8$ is non-zero. Finally, in each case we have the equation (1.3.3) holding. One checks that we can write this the function defining this equation projectively as

$$\begin{aligned} f \quad = \quad & i_8 i_{15}^2 + 4 i_8 i_4^2 + 4 i_8 i_6^2 + 4 i_8 i_7^2 + 4 i_8 i_{10}^2 + 4 i_8 i_{12}^2 + 4 i_8 i_{14}^2 \\ & -4 i_8 i_3 i_{12} - 4 i_8 i_2 i_{10} - 4 i_8 i_5 i_{14} + 4 i_{10} i_{12} i_{14} \\ & -i_8 i_{12}^2 - i_8 i_{13}^2 - i_8 i_{10}^2 \\ & +2 i_8 i_1 i_{13} + 2 i_8 i_1 i_9 + 2 i_8 i_1 i_{11} \\ & -2 i_8 i_{10} i_{12} - 2 i_8 i_{10} i_{14} - 2 i_8 i_{12} i_{14} \\ & -i_8 i_1^2 - 16 i_8^3, \end{aligned}$$

so finally we conclude that ϕ is a map

$$\phi{:}(Q \times Q \times Q)^{nn} \to Y'' := \{(i_1 : ... : i_{15}) \in Y' \,|\, (i_1 \neq 0 \vee ... \vee i_8 \neq 0) \wedge f = 0\}.$$

Finally, this motivates our claim:

PROPOSITION 25. *The map $\phi : (Q \times Q \times Q)^{nn} \to Y''$, as defined above, is a good quotient.*

2.6.2. Glueing together coordinate rings. Another approach to looking at the invariants on our affine cover is not to glue together the quotients, but to glue together the coordinate rings. The idea is as follows.

Given U_α and U_β , the coordinate ring $\mathbb{C}[U_\alpha \cup U_\beta]$ can be found by looking at functions that are in the one coordinate ring and but not in the other, and then adding them. Thus, $\mathbb{C}[U_\alpha \cup U_\beta]$ is described as an isomorphism between exensions of the coordinate rings of U_α and U_β (the so-called *glueing together* of the coordinate rings). This construction then of course also gives rise to an isomorphism between extensions of $\mathbb{C}[U_\beta]^{SL_2}$ and $\mathbb{C}[U_\beta]^{SL_2}$ to give $\mathbb{C}[U_\alpha \cup U_\beta]^{SL_2}$. Thus, by considering the above cover, we can determine the algebra of invariants of, in this case, $(Q \times Q \times Q)^{ss}$.

2.6.2.1. *Glueing $U_{(0,0,0)}$ and $U_{(1,0,0)}$.* We now glue together two of the affine parts of our covering: $U_{(0,0,0)}$, corresponding to our usual space $SL_2 \times SL_2 \times SL_2$, and $U_{(1,0,0)}$. We use the above notation from section 2.4 for the coordinate rings, with the convention that we add a single apostrophe for $U_{(0,0,0)}$ and a two apostrophes for $U_{(1,0,0)}$.

$U_{(0,0,0)}$ then clearly has coordinate ring

$$\begin{aligned} \mathbb{C}[U_{(0,0,0)}] \quad = \quad & \mathbb{C}[a'_1, b'_1, c'_1, d'_1, a'_2, b'_2, c'_2, d'_2, a'_3, b'_3, c'_3, d'_3]/ \\ & \qquad (a'_1 d'_1 - b'_1 c'_1 = 1, a'_2 d'_2 - b'_2 c'_2 = 1, a'_3 d'_3 - b'_3 c'_3 = 1) \end{aligned}$$

and algebra of invariants

$$\mathbb{C}[\mathrm{Tr} y'_1, \mathrm{Tr} y'_2, \mathrm{Tr} y'_3, \mathrm{Tr} y'_1 y'_2, \mathrm{Tr} y'_1 y'_3, \mathrm{Tr} y'_2 y'_3, \mathrm{Tr}(y'_1 y'_2 y'_3 - y'_1 y'_3 y'_2)]/(f),$$

with f from (1.3.3).

For $U_{(1,0,0)}$, we get coordinate ring

$$\begin{aligned}\mathbb{C}[U_{(1,0,0)}] \quad = \quad & \mathbb{C}[a_1'', b_1'', c_1'', d_1'', a_2'', b_2'', c_2'', d_2'', a_3'', b_3'', c_3'', d_3'']/ \\ & (a_1'' - a_1''^2 - b_1''c_1'' = d_1''^2, a_2'd_2' - b_2'c_2' = 1, a_3'd_3' - b_3'c_3' = 1),\end{aligned}$$

and algebra of invariants $\mathbb{C}[U_{(1,0,0)}]^{SL_2}$ equal to

$$\mathbb{C}[d_1'', \mathrm{Tr}y_2'', \mathrm{Tr}y_3'', \mathrm{Tr}y_1''y_2'', \mathrm{Tr}y_1y_3'', \mathrm{Tr}y_2''y_3'', \mathrm{Tr}(y_1''y_2''y_3'' - y_1''y_3''y_2'')]/(f''),$$

with f'' again from (1.3.3).

We now calculate $\mathbb{C}[U_{(0,0,0)} \cup U_{(1,0,0)}]$: this is

$$\begin{aligned}\mathbb{C}[a_1', b_1', c_1', d_1', a_2', ..., d_3', \frac{1}{a_1' + d_1'}]/(I') \quad & \cong \quad \mathbb{C}[a_1'', b_1'', c_1'', z_1'', a_2'', ..., d_3'', \frac{1}{d_1''}]/(I'') \\ a_2', ..., d_3' \quad & \leftrightarrow \quad a_2'', ..., d_3'' \\ a_1', b_1', c_1', d_1' \quad & \mapsto \quad \frac{a_1''}{d_1''}, \frac{b_1''}{d_1''}, \frac{c_1''}{d_1''}, \frac{1 - a_1''}{d_1''} \\ \frac{1}{a_1' + d_1'} \quad & \leftrightarrow \quad d_1'' \\ a_1'', b_1'', c_1'' \quad & \mapsto \quad \frac{a_1'}{a_1' + d_1'}, \frac{b_1'}{a_1' + d_1'}, \frac{c_1'}{a_1' + d_1'}.\end{aligned}$$

This gives rise to the following algebra of invariants $\mathbb{C}[U_{(0,0,0)} \cup U_{(1,0,0)}]^{SL_2}$:

$$\begin{aligned}\mathbb{C}[\mathrm{Tr}y_1', ..., \frac{1}{\mathrm{Tr}y_1'}]/(...) \quad & \cong \quad \mathbb{C}[d_1'', ..., \frac{1}{d_1''}]/(...) \\ \mathrm{Tr}y_1' \quad & \leftrightarrow \quad \frac{1}{d_1''} \\ \mathrm{Tr}y_2', \mathrm{Tr}y_3' \quad & \leftrightarrow \quad \mathrm{Tr}y_2'', \mathrm{Tr}y_3'' \\ \mathrm{Tr}y_1'y_2', \mathrm{Tr}y_1'y_3', \mathrm{Tr}y_2'y_3' \quad & \mapsto \quad \mathrm{Tr}y_1''y_2'' \cdot \frac{1}{d_1''}, \mathrm{Tr}y_1y_3'' \cdot \frac{1}{d_1''}, \mathrm{Tr}y_2''y_3'' \\ \mathrm{Tr}(y_1'y_2'y_3' - y_1'y_3'y_2') \quad & \mapsto \quad \mathrm{Tr}(y_1''y_2''y_3'' - y_1''y_3''y_2'') \cdot \frac{1}{d_1''} \\ \mathrm{Tr}y_1''y_2'', \mathrm{Tr}y_1y_3'', \mathrm{Tr}y_2''y_3'' \quad & \mapsto \quad \mathrm{Tr}y_1'y_2' \cdot \frac{1}{\mathrm{Tr}y_1'}, \mathrm{Tr}y_1'y_3' \cdot \frac{1}{\mathrm{Tr}y_3'}, \mathrm{Tr}y_2'y_3' \\ \mathrm{Tr}(y_1''y_2''y_3'' - y_1''y_3''y_2'') \quad & \mapsto \quad \mathrm{Tr}(y_1'y_2'y_3' - y_1'y_3'y_2') \cdot \frac{1}{\mathrm{Tr}y_1'}\end{aligned}$$

Here, (...) stands for the obvious set of relations: the usual variant of (1.3.3) and the relations $\mathrm{Tr}y_1' \cdot \frac{1}{\mathrm{Tr}y_1'} = 1$ etcetera.

2.6.2.2. *Glueing together $U_{(0,0,0)}$ and $U_{(1,1,1)}$.* As a second and last example before attempting to tackle the total cover, we glue together $U_{(0,0,0)}$ and $U_{(1,1,1)}$. We have coordinate ring

$$\begin{aligned}\mathbb{C}[U_{(1,1,1)}] \quad = \quad & \mathbb{C}[a_1'', b_1'', c_1'', d_1'', a_2'', b_2'', c_2'', d_2'', a_3'', b_3'', c_3'', d_3'']/ \\ & (a_1'' - a_1''^2 - b_1''c_1'' = d_1''^2, a_2'' - a_2''^2 - b_2''c_2'' = d_2''^2, \\ & a_3'' - a_3''^2 - b_3''c_3'' = d_3''^2),\end{aligned}$$

and algebra of invariants $\mathbb{C}[U_{(1,1,1)}]^{SL_2}$ equal to:

$$\mathbb{C}[d_1'', d_2'', d_3'', \mathrm{Tr}y_1''y_2'', \mathrm{Tr}y_1y_3'', \mathrm{Tr}y_2''y_3'', \mathrm{Tr}(y_1''y_2''y_3'' - y_1''y_3''y_2'')]/(f'')$$

for some variant f'' of (1.3.3).

The coordinate ring $\mathbb{C}[U_{(0,0,0)} \cup U_{(1,1,1)}]$ is as follows:

$$\begin{array}{rcl}
\mathbb{C}[a_1', ..., d_3', \dfrac{1}{a_1'+d_1'}, \dfrac{1}{a_2'+d_2'} \dfrac{1}{a_3'+d_3'}]/(I') & \cong & \mathbb{C}[a_1'', ..., d_3'', \dfrac{1}{d_1''}, \dfrac{1}{d_2''}, \dfrac{1}{d_3''}]/(I'') \\
a_i', b_i', c_i', d_i' & \mapsto & \dfrac{a_i''}{d_i''}, \dfrac{b_i''}{d_i''}, \dfrac{c_i''}{d_i''}, \dfrac{1-a_i''}{d_i''} \\
\dfrac{1}{a_i'+d_i'} & \leftrightarrow & d_i'' \\
a_i'', b_i'', c_i'' & \mapsto & \dfrac{a_i'}{a_i'+d_i'}, \dfrac{b_i'}{a_i'+d_i'}, \dfrac{c_i'}{a_i'+d_i'}.
\end{array}$$

The two isomorphic forms of $\mathbb{C}[U_{(0,0,0)} \cup U_{(1,1,1)}]^{SL_2}$ this gives rise to are:

$$\begin{array}{rcl}
\mathbb{C}[\mathrm{Tr}y_1', ..., \dfrac{1}{\mathrm{Tr}y_1'}, \dfrac{1}{\mathrm{Tr}y_2'}, \dfrac{1}{\mathrm{Tr}y_3'}]/(...) & \cong & \mathbb{C}[z_1'', ..., \dfrac{1}{d_1''}, \dfrac{1}{d_2''}, \dfrac{1}{d_3''}]/(...) \\
\mathrm{Tr}y_i' & \leftrightarrow & \dfrac{1}{d_i''} \\
\mathrm{Tr}y_i'y_j' & \mapsto & \mathrm{Tr}y_i''y_j'' \cdot \dfrac{1}{d_i''} \cdot \dfrac{1}{d_j''} \\
\mathrm{Tr}(y_1'y_2'y_3' - y_1'y_3'y_2') & \mapsto & \mathrm{Tr}(y_1''y_2''y_3'' - y_1''y_3''y_2'') \cdot \dfrac{1}{d_1''} \cdot \dfrac{1}{d_2''} \cdot \dfrac{1}{d_3''} \\
\mathrm{Tr}y_i''y_j'' & \mapsto & \mathrm{Tr}y_i'y_j' \cdot \dfrac{1}{\mathrm{Tr}y_i'} \cdot \dfrac{1}{\mathrm{Tr}y_j'}, \\
\mathrm{Tr}(y_1''y_2''y_3'' - y_1''y_3''y_2'') & \mapsto & \mathrm{Tr}(y_1'y_2'y_3' - y_1'y_3'y_2') \cdot \dfrac{1}{\mathrm{Tr}y_1'} \cdot \dfrac{1}{\mathrm{Tr}y_2'} \cdot \dfrac{1}{\mathrm{Tr}y_3'}.
\end{array}$$

2.6.2.3. *The complete cover.* Now let us see what happens if, for instance, we want to determine the coordinate ring of

$$\mathbb{C}[U_{(0,0,0)} \cup U_{(1,1,1)} \cup U_{(1,0,0)}].$$

We do this by glueing together the coordinate rings

$$\mathbb{C}[\mathrm{Tr}y_1', ..., \frac{1}{\mathrm{Tr}y_1'}, \frac{1}{\mathrm{Tr}y_2'}, \frac{1}{\mathrm{Tr}y_3'}]/(...),$$

which is the coordinate ring $\mathbb{C}[U_{(0,0,0)} \cup U_{(1,1,1)}]$ seen as an extension of $\mathbb{C}[U_{(0,0,0)}]$, and the coordinate ring

$$\begin{array}{rcl}
\mathbb{C}[U_{(1,0,0)}] & = & \mathbb{C}[a_1'', b_1'', c_1'', d_1'', a_2'', b_2'', c_2'', d_2'', a_3'', b_3'', c_3'', d_3'']/ \\
 & & (a_1'' - a_1''^2 - b_1''c_1'' = d_1''^2, a_2'd_2' - b_2'c_2' = 1, a_3'd_3' - b_3'c_3' = 1).
\end{array}$$

But then to glue these rings together, it suffices to add the new coordinates

$$\frac{1}{a_2''+d_2''}, \frac{1}{a_3''+d_3''}$$

to the coordinate ring of $\mathbb{C}[U_{(0,0,0)}]$, and the coordinate ring of $\mathbb{C}[U_{(0,0,0)} \cup U_{(1,1,1)}]$ remains unchanged. We summarize this result in the following claim:

PROPOSITION 26. *The coordinate ring of* $\mathbb{Q}[(Q \times Q \times Q)^{nn}]$ *is:*

$$\mathbb{C}[a_1', b_1', c_1', d_1', a_2', b_2', c_2', d_2', a_3', b_3', c_3', d_3', \frac{1}{a_1'+d_1'}, \frac{1}{a_2'+d_2'} \frac{1}{a_3'+d_3'}];$$

its algebra of invariants is

$$\begin{array}{l}
\mathbb{C}[\mathit{Tr}y_1, \mathit{Tr}y_2, \mathit{Tr}y_3, \mathit{Tr}y_1y_2, \mathit{Tr}y_1y_3, \mathit{Tr}y_2y_3, \mathit{Tr}(y_1y_2y_3 - y_1y_3y_2), \\
\dfrac{1}{\mathit{Tr}y_1}, \dfrac{1}{\mathit{Tr}y_2}, \dfrac{1}{\mathit{Tr}y_3}]/(I),
\end{array}$$

where

$$(I) = \left(f,\, Try_1 \cdot \frac{1}{Try_1} = 1,\, Try_2 \cdot \frac{1}{Try_2} = 1,\, Try_3 \cdot \frac{1}{Try_3} = 1\right);$$

f as in (1.3.3).

Even though we now have a form for the algebra of invariants, it is now not obvious whether we can turn this into a good quotient. At least it does not seem obvious how to see that the good quotient from Proposition 25 follows from this description of the algebra of invariants.

Also, it is not obvious how to perhaps construct another projective map from this description. For example, $\mathrm{Tr}y_1$ as a function on $Q \times Q \times Q$ is

$$\frac{\mathrm{Tr}y_1\Delta_2\Delta_3}{\Delta_1\Delta_2\Delta_3},$$

but then $(\mathrm{Tr}y_1)^{-1}$ is

$$\frac{\Delta_1\Delta_2\Delta_3}{\mathrm{Tr}y_1\Delta_2\Delta_3},$$

so then an obvious projective map would be writing every invariant as a quadratic fraction in $Q \times Q \times Q$ and then turning this into a projective map as follows:

$$\begin{aligned}((M_1 : \Delta_1), \ldots, (M_3 : \Delta_3)) \quad \mapsto \quad & (\mathrm{Tr}y_1^2\Delta_2\mathrm{Tr}y_2\Delta_3\mathrm{Tr}y_3 : \mathrm{Tr}y_1\Delta_1\mathrm{Tr}y_2^2\Delta_3\mathrm{Tr}y_3 : \\ & \mathrm{Tr}y_1\Delta_1\mathrm{Tr}y_2\Delta_2\mathrm{Tr}y_3^2 : \mathrm{Tr}y_1y_2\mathrm{Tr}y_1\mathrm{Tr}y_2\mathrm{Tr}y_3\Delta_3 : \\ & \mathrm{Tr}y_1y_3\mathrm{Tr}y_1\mathrm{Tr}y_3\mathrm{Tr}y_2\Delta_2 : \mathrm{Tr}y_2y_3\mathrm{Tr}y_2\mathrm{Tr}y_3\mathrm{Tr}y_1\Delta_1 : \\ & \mathrm{Tr}(y_1y_2y_3 - y_1y_3y_2)\mathrm{Tr}y_1\mathrm{Tr}y_2\mathrm{Tr}y_3 : \\ & \Delta_1^2\mathrm{Tr}y_2\Delta_2\mathrm{Tr}y_3\Delta_3 : \Delta_2^2\mathrm{Tr}y_1\Delta_1\mathrm{Tr}y_3\Delta_3 : \\ & \Delta_3^2\mathrm{Tr}y_1\Delta_1\mathrm{Tr}y_2\Delta_2 : \mathrm{Tr}y_1\Delta_1\mathrm{Tr}y_2\Delta_2\mathrm{Tr}y_3\Delta_3).\end{aligned}$$

But note, for example, that this function is already not defined whenever $\mathrm{Tr}y_i = 0$ for some i and $\Delta_j = 0$ for some j so there is little hope we can turn this into a quotient for the whole of $(Q \times Q \times Q)^{nn}$.

We conclude that first constructing good quotients and then glueing these together rather than the other way round seems to be the better method. (Indeed, this former method is used for the theoretical foundation of geometric invariant theory.)

CHAPTER 3

Stability, semi-stability and quotients

In the previous chapter, we introduced an embedding of $\mathbb{C}[SL_2^m]$ into a projective space, and constructed a cover of most of this space for which we were able to construct a good quotient. As we noted, however, this cover was not complete in the sense that there were affine parts that were not covered. Also, the quotient on our cover was not a geometric quotient.

In this chapter we will look at both aspects. First, in sections 3.1 and 3.2 we introduce some general theory which tells us when we can make good and geometric quotients; next in section 3.3 we see how how the theory applies to the case of a SL_2-action. In the following sections, we perform calculations for the cases $m = 1$, $m = 2$ and $m = 3$ and find some results for general m.

3.1. Semi-stable and stable points

We want to know for which points of a projective variety X we can construct quotients. For this, GIT introduces the following definitions:

Definition 27. Let V be a G-representation and $X \hookrightarrow \mathbb{P}(V)$ be a projective variety so that G acts linearly on X. Let be $\hat{X} \hookrightarrow V$ the corresponding affine variety so that we have the canonical map

$$\hat{X}\backslash\{0\} \overset{\phi}{\to} X : x \overset{\phi}{\mapsto} [x].$$

A point $x \in X$ is then called *semi-stable* if there is a point $\hat{x} \in \hat{X}$; $\phi(\hat{x}) = x$ (a *lift* of x) so that $0 \notin \overline{G.\hat{x}}$. Similarly, x is called *stable* if there is an $\hat{x}$ so that $G \cdot \hat{x}$ is closed, and $\dim(G \cdot \hat{x}) = \dim G$. A point is called *unstable* if it is not semi-stable. (Note that unstable is *not* the opposite of stable!)

Denote $X^{ss} = \{x \in X \mid x \,\text{is semi-stable}\}$ and $X^s = \{x \in X \mid x \,\text{is stable}\}$.

Then the following holds:

Theorem 28. *Let X be a projective variety. Then X^{ss} admits a good quotient ϕ. Also, $\phi : X^s \to \phi(X^s)$ is a geometric quotient.*

In [**9**, Lemma 6.4.5] it is shown that in the projective case, a point $x \in X$ being semi-stable is equivalent to the existence of a homogeneous G-invariant polynomial with strictly positive degree which does not vanish at x. Note that this corresponds exactly to the existence of an open, affine, G-stable set containing x, which corresponds to our previous discussion on coverings.

Now suppose we have a good quotient not for the whole of X^{ss}, but for some part of it. The following Corollary shows that we still have a geometric quotient:

Proposition 29. *Let X be a projective variety, and suppose we have $X^s \subset X' \subset X^{ss}$, where for some set S of homogeneous, G-invariant polynomials with strictly positive degree,*

$$X' = \bigcup_{f\in S} X_f;\ X_f = \{x \in X \mid f(x) \neq 0\}.$$

If (Y, ϕ) is a good quotient for X', then $\phi : X^s \to \phi(X^s)$ is a geometric quotient.

Proof. Obviously for any affine part X_f, the good quotient of X' must be isomorphic to the good quotient of X^{ss}, so glueing these good quotients together, the good quotient for X^{ss} restricted to the set X' is isomorphic to the good quotient of X'. But since $X^s \subset X'$, then by the previous theorem $\phi : X^s \to \phi(X^s)$ is a geometric quotient. □

3.2. Mumford's criterion

To determine whether points in a projective variety are (semi-)stable, one can use the Hilbert-Mumford criterion. This criterion of formulated by looking at the action of so-called "one-parameter subgroups" of an algebraic group G; these are simply algebraic group morphisms $\lambda : \mathbb{C}^* \to G$. We suppose that G acts linearly on X.

Suppose we have a projective variety $X \subset \mathbb{P}(V)$, corresponding affine variety $\hat{X} \subset V$ and canonical map ϕ, as above. Let $x = \phi(\hat{x})$ and λ be given. In [**8**] it is shown that by choosing a basis $\{e_i\}$ for V suitably, we can write the action of λ on V as:

$$\lambda(t) \cdot \hat{x} = \sum c_i t^{r_i} e_i.$$

The *weights* of x with respect to λ are now defined to be the $\{r_i \,|\, c_i \neq 0\}$. We can now formulate the criterion as follows:

Theorem 30. *(Hilbert-Mumford) In the above setting, x is semi-stable if and only if for every λ, x has at least one non-positive weight; x is stable if and only if for every non-trivial λ, it has at least one negative weight.*

3.3. Our case

Let is now try to apply this theory to our normal projective situation in which we project every matrix in one $\mathbb{P}^4$. Let

$$Q = \left\{(a : b : c : d : \Delta) \,|\, ad - bc = \Delta^2\right\} \subset \mathbb{P}^4$$

be the embedding of one such matrix. We are then interested in finding out the quotient of a product of several Q's.

For this, we will need to look at one-parameter subgroups of SL_2. The following lemma helps:

Lemma 31. *Let $\lambda(t)$ be a non-trivial one-parameter subgroup of SL_2. Then there is a non-trivial one-parameter subgroup $\tilde{\lambda}$ and a similarity matrix Q so that $\tilde{\lambda}(t) = Q^{-1} diag(t, t^{-1}) Q$, and λ and $\tilde{\lambda}$ have the same stable and semi-stable points.*

Proof. Let us take a one-parameter subgroup $\lambda(t)$ of SL_2, which is by definition an algebraic group morphism $\lambda : \mathbb{C}^* \to SL_2$. Because λ is non-trivial and algebraic, Kerλ is an algebraic subgroup Ker$\lambda \subsetneq \mathbb{C}^*$, so it must be a finite group of nth roots of unity, the cyclic group C_n. By letting $\tilde{\lambda}(z) = \lambda(z^n)$, we see that then $\tilde{\lambda}$ is an isomorphism $\mathbb{C}^* \to \lambda(\mathbb{C}^*)$:

$$\begin{tikzcd}
\mathbb{C}^*/C_n \arrow[r, "\lambda"] \arrow[d, "z \mapsto z^n"'] & SL_2 \\
\mathbb{C}^*, \arrow[ur, dashed, "\tilde{\lambda}"'] &
\end{tikzcd}$$

Because clearly λ and $\tilde{\lambda}$ have the same stable and semi-stable points (for a given point, $\tilde{\lambda}$ has the same weights as λ multiplied by n), without loss of generality we may assume λ is an isomorphism $\mathbb{C}^* \to \lambda(\mathbb{C}^*)$.

Now suppose that some $A \in \lambda(\mathbb{C}^*)$ is not diagonalizable, then in Jordan form either $A \in \lambda(\mathbb{C}^*)$ or $-A \in \lambda(\mathbb{C}^*)$ can be written as

$$\hat{A} = Q^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Q$$

using some similarity matrix Q. But then also

$$\hat{A}^n = Q^{-1} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} Q \in \lambda(\mathbb{C}^*) \;\; \forall n \in \mathbb{N}.$$

Note that this set is Zariski-dense in the affine variety

$$V = \left\{ Q^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} Q \,|\, x \in \mathbb{C} \right\},$$

because for any closed set $Z(f_1, ..., f_n) \subsetneq V$, each $f_n \in k[V]$ can have only finitely many values for x for which it is zero. Because $\lambda(\mathbb{C}^*)$ is closed as the image of a closed set, then certainly $V \subset \lambda(\mathbb{C}^*)$. But this means we have an infinite additive algebraic subgroup of $\mathbb{C}^*$, which is impossible.

So in fact, all matrices in $\lambda(\mathbb{C}^*)$ are diagonalizable, and because it is a commutative group, they are in fact simultaneously diagonalizable. For suppose v is an eigenvector of a matrix A so that $Av = \mu v$ for some V, then $A(Bv) = B(Av) = B(\lambda v) = \lambda Bv$, so $B(v)$ is an eigenvector of A corresponding to the same eigenvalue λ, so $Bv = \mu v$ for some μ. (This assumes that A does not have a double eigenvalue, but in this case it is $* \cdot \mathrm{Id}$ which is invariant under conjugation.)

By the same Zariski arguments, we then have that for some similarity matrix Q,

$$\lambda(\mathbb{C}^*) = \left\{ Q \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} Q^{-1} \,|\, x \neq 0 \right\} \cong \mathbb{C}^*.$$

Thus $\lambda(t) = Q^{-1}\mathrm{diag}(t, t^{-1})Q$ or $\lambda(t) = Q^{-1}\mathrm{diag}(t^{-1}, t)Q$ because we assumed λ to be an isomorphism $\mathbb{C}^* \to \lambda(\mathbb{C}^*)$. In the first case, we are done; in the second case, using another similarity matrix Q that switches the two eigenvalues gives us the desired conclusion as well. $\square$

We also introduce the following notion:

Definition 32. (See [**4**, p.18]) A matrix tuple $(A_1, A_2, ..., A_n) \in M^k_{2\times 2}$ is called *reducible* if and only if it can be simultaneously conjugated to the form

$$\left(\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, ..., \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right);$$

it is called *irreducible* otherwise.

3.4. One matrix

First of all, let us look at the situation when we have just one matrix. The action on Q (conjugating the matrix formed by a, b, c, d and leaving Δ intact) extends to an action on the whole of $\mathbb{P}^4$, so we can consider its semi-stable points.

Let us apply Mumford's criterion to find unstable points. For a matrix M to be unstable, there must be a one-parameter subgroup such that all weights of M are positive. By Lemma 31 we can assume that M acts diagonally, so after some basis transformation, the action of SL_2 is:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \Delta \right) \mapsto \left(\begin{pmatrix} a & t^2 b \\ t^{-2} c & d \end{pmatrix}, \Delta \right).$$

By the criterion any unstable point must now be in the span of the single basis vector$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. In other words, the unstable points are exactly (the single orbit of) the nilpotent matrices.

We conclude that the good quotient for Q^{nn} constructed earlier already already covers all semi-stable points. So, let

$$\begin{aligned}\phi : Q^{ss} &\to \mathbb{P}^1 \\ (M : \Delta) &\mapsto (\mathrm{tr}M : \Delta).\end{aligned} \tag{3.4.1}$$

Note that indeed the result is not defined exactly when $\mathrm{Tr}M = \det M = 0$, ie when M is nilpotent. Thus:

PROPOSITION 33. *Let $Y = \mathbb{P}^1$, ϕ as in (3.4.1). Then (Y, ϕ) is a good quotient for Q^{ss}.*

We also apply Mumford's criterion to find stable points. We find that for a matrix M not to be stable, there has to be a λ such that all weights are non-negative. This means that conjugated suitably, we must have $M_{21} = 0$. But of course this holds for any matrix: just write it in Jordan form.

This is not so strange when one considers the definition of points being stable: for this we need that the dimension (in affine space) of the orbit of the matrix equals $\dim(SL_2) = 3$. But conjugation leaves the trace and matrix of the matrix intact, so this can never be the case.

3.5. Mumford's criterion for $Q \times ... \times Q$

We now find a criterium for (semi-)stability in a k-fold product of Q's, with $k > 2$. In this case, we get an embedding

$$\begin{aligned} P &= Q \times ... \times Q \\ &= \oplus_{i=1,..k} \left\{ (a_i : b_i : c_i : d_i : \Delta_i) \,|\, a_i d_i - b_i c_i = \Delta_i^2 \right\} \\ &\subset \mathbb{P}^4 \times ... \times \mathbb{P}^4 \hookrightarrow \mathbb{P}^{5^k - 1} \end{aligned}$$

with basis consisting of k-fold products of $a_i, b_i, c_i, d_i, \Delta_i$'s for $i = 1, ..., k$. The criterion we get is slightly awkward. First we calculate the unstable points:

LEMMA 34. *A point $x = (M_1 : \Delta_1, ..., M_m : \Delta_m) \in Q \times ... \times Q$ is unstable if and only if for some i, matrix M_i is nilpotent and when x is conjugated so that M_i is in Jordan form, the number of matrices with bottom-left element non-zero is smaller than the number of nilpotent matrices commuting with M_i (including M_i).*

PROOF. A point x is unstable if there is a one-parameter subgroup λ, which we can assume to acts as $\mathrm{diag}(\lambda, \lambda^{-1})$, so that all weights with respect to λ are strictly positive. Then for any combination of a, c, d, Δ,

$$\prod_{j=1,..,k} \{a_j, c_j, d_j, \Delta_j\} = 0.$$

But this means that at least one matrix M_i is nilpotent: for if we could choose an $a_i, c_i, d_i, \Delta_i \neq 0$ for any i, then its product would be non-zero as well. So we can conjugate x so that M_i is in Jordan form.

But then consider the $(k-1)$-tuple $\hat{x}$ we get if we remove M_i from x: then x is unstable exactly when all weights of $\hat{x}$ with respect to λ must be non-negative. This is automatically satisfied if all $c_j = 0$. On the other hand, if $c_j \neq 0$ for some j, then consider the $(k-2)$-tuple $\tilde{x}$ we get if we remove M_i, M_j from x. Then x is unstable exactly when all weights of $\tilde{x}$ with respect to λ are strictly positive.

But this is the case we started with, so inductively we obtain the statement we want to prove. □

Then directly:

PROPOSITION 35. *A point $x = (M_1 : \Delta_1, ..., M_m : \Delta_m) \in Q \times ... \times Q$ is semi-stable if for any of its nilpotent matrices M_i, when we conjugate x so that M_i is in Jordan form, the number of matrices with bottom-left element non-zero is at least the number of nilpotent matrices commuting with M_i (including M_i).*

For stable points the situation is a bit easier:

PROPOSITION 36. *A point $x = (M_1 : \Delta_1, ..., M_m : \Delta_m) \in Q \times ... \times Q$ is stable if and only if it is irreducible and none of the matrices is nilpotent.*

PROOF. Such a tuple is not stable if and only if there is one-parameter subgroup λ so that all weights with respect to λ are ≥ 0. First suppose we have a matrix tuple $(M_1, ..., M_m)$ that is not stable. Using the above notation of $M_i = (a_i : b_i : c_i : d_i : \Delta_i)$, we get that

$$c_i \cdot \prod_{j \neq i} \{a_j, c_j, d_j, \Delta_j\} = 0$$

So suppose we have some $c_i \neq 0$. Then for one j we must have $a_j = c_j = d_j = \Delta_j = 0$; for suppose for all j we that one these four entries is non-zero, then the product of all these choices would be non-zero as well. On the other hand, all $c_i = 0$ gives a reducible pair. Clearly the other direction of the proposition also holds. ☐

Comparing the cover from Chapter 2 with these results, we see:

$$(Q \times ... \times Q)^s \subsetneq (Q \times ... \times Q)^{nn} \subsetneq (Q \times ... \times Q)^{ss}.$$

By Proposition 29 we now directly conclude:

COROLLARY 37. *The good quotient (Y, ϕ) for $(Q \times ... \times Q)^{nn}$ constructed in Chapter 2 is a geometric quotient $(Q \times ... \times Q)^s \to \phi((Q \times ... \times Q)^s)$.*

3.6. The case $m = 2$

Let us now look at the situation for $Q \times Q$. The criterion for stability above can be formulated as follows:

COROLLARY 38. *A point $(M_1 : \Delta_1, M_2 : \Delta_2) \in Q \times Q$ is semi-stable if and only if it is irreducible or neither matrix is nilpotent.*

As noted before, the subset$(Q \times Q)^{nn}$ for which we constructed a good quotient in Proposition 24 satisfies:

$$(Q \times Q)^s \subsetneq (Q \times Q)^{nn} \subsetneq (Q \times Q)^{ss}.$$

Here, $(Q \times Q)^{nn}$ had quotient space

$$Y = \left\{(x_1 : x_2 : x_3 : x_4 : x_5) \in \mathbb{P}^4 \,|\, x_1 x_2 = x_3 x_4\right\} \backslash (0 : 0 : 0 : 0 : 1),$$

with the quotient map ϕ from (2.5.1):

$$(M_1 : \Delta_1, M_2 : \Delta_2) \mapsto (\mathrm{Tr} M_1 \Delta_2 : \Delta_1 \mathrm{Tr} M_2 : \mathrm{Tr} M_1 \mathrm{Tr} M_2 : \Delta_1 \Delta_2 : \mathrm{Tr} M_1 M_2).$$

Note that $(Q \times Q)^{ss} \backslash (Q \times Q)^{nn}$ exactly corresponds to pairs of matrices (M_1, M_2) of which at least one is nilpotent, but $\mathrm{Tr} M_1 M_2 \neq 0$. Under this map ϕ, these points are sent exactly to the point $(0 : 0 : 0 : 0 : 1)$ we excluded: hence,

$$\phi : X^{ss} \to \left\{(x_1 : x_2 : x_3 : x_4 : x_5) \in \mathbb{P}^4 \,|\, x_1 x_2 = x_3 x_4\right\}$$

is a well-defined map. However, it may not be a quotient because it does not separate the various different orbits in X^{ss}.

We now also know that if we would know the structure of the algebra of invariants of the affine part defined by $\mathrm{Tr} M_1 M_2 \neq 0$, we would have covered the complete $(Q \times Q)^{ss}$. However, we also note that finding the structure of this algebra of invariants might not be so simple because our usual methods with module isomorphisms do not seem to work. Also, if ϕ is not already a good quotient then building such a quotient seems to boil down to "blowing up" the map ϕ at the image point $(0 : 0 : 0 : 0 : 1)$, of which it does not seem obvious how this can be done.

Now we look at how $(Q \times Q)^s$ behaves under ϕ. By Proposition 29, we know that restricted to $(Q \times Q)^s$, ϕ is a geometric quotient.

We now determine the image of ϕ restricted to $(Q \times Q)^s$. In [**4**, p.18] it is proved that a matrix pair (M_1, M_2) being irreducible is equivalent to $\mathrm{Tr}((M_1M_2)^2 - M_1^2M_2^2) \neq 0$. One checks that this expression can be written in terms of elements of the algebra of invariants as follows:

$$\begin{aligned}\mathrm{Tr}((M_1M_2)^2 - M_1^2M_2^2) &= (\mathrm{Tr}M_1)^2\Delta_2^2 + \Delta_1^2(\mathrm{Tr}M_2)^2 + (\mathrm{Tr}M_1M_2)^2 \\ &\quad -\mathrm{Tr}M_1\mathrm{Tr}M_2\mathrm{Tr}M_1M_2 - 4\Delta_1^2\Delta_2^2;\end{aligned}$$

see [**11**] for how we obtained this relation. We see that X^s maps to the open subset

$$Y' = \{(x_1 : x_2 : x_3 : x_4 : x_5) \in \mathbb{P}^5 \,|\, x_1^2 + x_2^2 + x_5^2 - x_3x_5 - 4x_4^2 \neq 0\} \subset Y.$$

But then we get the following result:

PROPOSITION 39. *Let Y' be the variety:*

$$\{(x_1 : x_2 : x_3 : x_4 : x_5) \,|\, x_1x_2 = x_3x_4,\ x_1^2 + x_2^2 + x_5^2 - x_3x_5 - 4x_4^2 \neq 0\} \subset \mathbb{P}^5$$

The map ϕ defined by (2.5.1) is a geometric quotient $X^s \to Y'$.

3.7. The case $m = 3$

We will now look at the case with 3 matrices. Recall that in the affine version of this case, apart from the usual degree 1 and degree 2 generators for the algebra of invariants, we have one degree 3 generator, and one relation holding between the generators.

In this case, our good quotient was the map $\phi{:}(Q \times Q \times Q)^{nn} \to \mathbb{P}^{14}$:

$$\begin{aligned}(M_1 : \Delta_1, ..., M_3 : \Delta_3) \overset{\phi}{\mapsto} &(\mathrm{Tr}M_1\mathrm{Tr}M_2\mathrm{Tr}M_3 : \mathrm{Tr}M_1\mathrm{Tr}M_2\Delta_3 : \mathrm{Tr}M_1\Delta_2\mathrm{Tr}M_3 : \\ &\mathrm{Tr}M_1\Delta_2\Delta_3 : \Delta_1\mathrm{Tr}M_2\mathrm{Tr}M_3 : \Delta_1\mathrm{Tr}M_2\Delta_3 : \\ &\Delta_1\Delta_2\mathrm{Tr}M_3 : \Delta_1\Delta_2\Delta_3 : \\ &\mathrm{Tr}M_1M_2\mathrm{Tr}M_3 : \mathrm{Tr}M_1M_2\Delta_3 : \\ &\mathrm{Tr}M_1M_3\mathrm{Tr}M_2 : \mathrm{Tr}M_1M_3\Delta_2 : \\ &\mathrm{Tr}M_2M_3\mathrm{Tr}M_1 : \mathrm{Tr}M_2M_3\Delta_1 : \\ &\mathrm{Tr}(M_1M_2M_3 - M_1M_3M_2)).\end{aligned}$$

The criterion for semi-stability in the $m = 3$ case is as follows:

COROLLARY 40. *(to Proposition 35) A point $(M_1 : \Delta_1, M_2 : \Delta_2, M_3 : \Delta_3)$ is unstable if at least one M_i is nilpotent and then all $M = M_j$ have $M_{21} = 0$, or two of the matrices are nilpotent and commuting.*

Now, it turns out that similarly to the case $m = 2$, a triple not being 0 on any of the above invariants is equivalent to it being semi-stable:

LEMMA 41. *A matrix triple $p = (M_1 : \Delta_1, M_2 : \Delta_2, M_3 : \Delta_3) \in Q \times Q \times Q$ is semi-stable if and only if the map ϕ above is well-defined in p.*

PROOF. First we note by direct calculation that all invariants are zero for the types of unstable triples described above. Conversely, suppose all invariant function are zero. We want to show that we have an unstable point.

Note that one of the three matrices has to be nilpotent because of the invariants $i_1, ..., i_8$: say it is matrix M_3. Notice that now directly all $i_1, ..., i_8$ are zero. We can now write M_3 in Jordan form so that $a_3 = c_3 = d_3 = \Delta_3 = 0$; $b_3 \neq 0$.

Also, one calculates that $\mathrm{Tr} M_1 M_3 = c_1 b_3$. Because $b_3 \neq 0$, if we look at i_{11} and i_{12}, then either M_2 is nilpotent or $c_1 = 0$. In the same way, either M_1 is nilpotent or $c_2 = 0$. If $c_1 = c_2 = 0$ then clearly we have an unstable point by our criterion.

Otherwise, suppose $c_1 \neq 0$, but M_2 is nilpotent and $c_2 = 0$. One calculates that

$$M_2 \cdot M_3 - M_3 \cdot M_2 = \begin{pmatrix} 0 & (d_2 - a_2) b_3 \\ 0 & 0 \end{pmatrix}; \ \mathrm{Tr} M_1 M_2 M_3 = c_1 (d_2 - a_2) b_3 = 0,$$

so in fact M_2 and M_3 commute and we have an unstable point by our criterion.

Finally, consider the case where $c_1, c_2 \neq 0$, and M_1 and M_2 are both nilpotent. We claim that M_1 and M_2 commute, ie,

$$C := M_1 \cdot M_2 - M_2 \cdot M_1 = \begin{pmatrix} -b_2 c_1 + b_1 c_2 & -2a_2 b_1 + 2a_1 b_2 \\ 2a_2 c_1 - 2a_1 c_2 & b_2 c_1 - b_1 c_2 \end{pmatrix} = 0.$$

The only invariant we can still use, i_{15}, gives us that $a_1 c_2 = a_2 c_1$, so $C_{21} = 0$. But nilpotency of M_1, M_2 gives $a_1^2 = b_1 c_1$ and $a_2^2 = b_2 c_2$, so

$$\begin{aligned} C_{11} &= -b_2 c_1 + b_1 c_2 = -\frac{a_2^2 c_1}{c_2} + \frac{a_1^2 c_2}{c_1} = \frac{a_1^2 c_2^2 - a_2^2 c_1^2}{c_1 c_2} \\ &= \frac{(a_1 c_2 - a_2 c_1)(a_1 c_2 + a_2 c_1)}{c_1 c_2} = 0; \\ C_{12} &= -2a_2 b_1 + 2a_1 b_2 = -2\frac{a_1 b_1 c_2}{c_1} + 2\frac{a_1 b_2 c_1}{c_1} = -2\frac{a_1}{c_1}(-b_2 c_1 + b_1 c_2) = 0. \end{aligned}$$

So in this case our criterion is also satisfied. This concludes our proof. ☐

This suggests that, again, the map extends to a map for the whole of $(Q \times Q \times Q)^{ss}$; in the terminology of Section 2.6, this suggests that possibly ϕ is a good quotient

$$\phi{:}(Q \times Q \times Q)^{ss} \to Y'\{(i_1 : ... : i_{15}) \in Y' \mid f = 0\}.$$

Again, this may be troublesome to prove.

Also, one could capculate the image under ϕ of $(Q \times Q \times Q)^s$ to find the form of the orbit space of the stable points by noting the following result from [**4**]:

PROPOSITION 42. [**4**, p.20] *A matrix triple* $(A, B, C) \in M_{2\times 2}^m$ *is reducible if and only if* (A, B)*,* (B, C) *and* (A, C) *are reducible and* $Tr(ABC - ACB) = 0$*.*

Thus, the subset $\phi((Q \times Q \times Q)^s) \subset \phi((Q \times Q \times Q)^{nn})$ would be defined by four equations, at least one of which should be non-zero: the three equations as before corresponding to the irreducibility of the (A, B), (B, C) and (B, C), and the equation $\mathrm{Tr}(ABC - ACB) \neq 0$.

3.8. Some notes on the general case

The above calculations for $m = 3$ should generalize to higher m without too much problems. To find the image of the stable points having found the image of the non-nilpotent points, one can use the following Corollary from [**4**]:

COROLLARY 43. [**4**, p.21] *A m-tuple* $(A_1, ..., A_m) \in M_{2\times 2}^m$ *is reducible if and only if each triple* $(A_{i_1}, A_{i_2}, A_{i_3})$ *taken from* $\{A_1, ..., A_m\}$ *is reducible.*

However, it is not clear whether the proof of Lemma 41 easily generalizes to more matrices, or whether it is even true at all. Thus we cannot say whether the map the quotient for $(Q \times ... \times Q)^{ss}$ would be defined, or indeed, a good quotient for the semi-stable points.

3.9. Interpreting stability for SL_2 matrices

We now do some calculations on orbits of SL_2 matrices in the affine coordinate ring, and compare this to our above results.

First, we look at the 3 matrix-case. Corollary 18 tells us that $\mathbb{C}[SL_2 \times SL_2 \times SL_2]^{SL_2}$ is a quotient ring with 7 generators a, b, c, d, e, f, g and one relation, $r \in \mathbb{C}[a, b, c, d, e, f, g]$ – see equation (1.3.3). The *singular locus* of this quotient is the set of points p in the corresponding algebraic variety where

$$\mathrm{grad}_p r = \left[\frac{\partial r}{\partial a}(p), ..., \frac{\partial r}{\partial g}(p)\right] = 0;$$

these are thus the points in the variety where no tangent plane can be defined. We study the relation between points in the singular locus and unstable points in the corresponding projective embedding above.

One can calculate the singular locus of the quotient using the Macaulay software package[1]. Using our familiar shorthands $tr_i = TrM_i$, $tr_{ij} = TrM_iM_j$, $tr_{ijk} = Tr(M_iM_jM_k - M_iM_kM_j)$, we get that the singular locus is the following ideal (I):

$$\begin{aligned}
(I) \quad = \quad & (\mathrm{tr}_{123}, \\
& -\mathrm{tr}_1\mathrm{tr}_2\mathrm{tr}_{12} + \mathrm{tr}_1^2 + \mathrm{tr}_2^2 + \mathrm{tr}_{12}^2 - 4, \\
& -\mathrm{tr}_2\mathrm{tr}_3\mathrm{tr}_{23} + \mathrm{tr}_2^2 + \mathrm{tr}_3^2 + \mathrm{tr}_{23}^2 - 4, \\
& -\mathrm{tr}_1\mathrm{tr}_3\mathrm{tr}_{13} + \mathrm{tr}_1^2 + \mathrm{tr}_3^2 + \mathrm{tr}_{13}^2 - 4, \\
& -\mathrm{tr}_1^2\mathrm{tr}_3 + \mathrm{tr}_2^2\mathrm{tr}_3 - \mathrm{tr}_2\mathrm{tr}_{12}\mathrm{tr}_{13} + \mathrm{tr}_1\mathrm{tr}_{12}\mathrm{tr}_{23} + 2\mathrm{tr}_1\mathrm{tr}_{13} - 2\mathrm{tr}_2\mathrm{tr}_{23}, \\
& -\mathrm{tr}_1\mathrm{tr}_2^2 + \mathrm{tr}_1\mathrm{tr}_3^2 - \mathrm{tr}_3\mathrm{tr}_{12}\mathrm{tr}_{23} + \mathrm{tr}_2\mathrm{tr}_{13}\mathrm{tr}_{23} + 2\mathrm{tr}_2\mathrm{tr}_{12} - 2\mathrm{tr}_3\mathrm{tr}_{13}, \\
& -\mathrm{tr}_1^2\mathrm{tr}_2 + \mathrm{tr}_2\mathrm{tr}_3^2 - \mathrm{tr}_3\mathrm{tr}_{12}\mathrm{tr}_{13} + \mathrm{tr}_1\mathrm{tr}_{13}\mathrm{tr}_{23} + 2\mathrm{tr}_1\mathrm{tr}_{12} - 2\mathrm{tr}_3\mathrm{tr}_{23}, \\
& \mathrm{tr}_1\mathrm{tr}_2^2\mathrm{tr}_3 - \mathrm{tr}_2\mathrm{tr}_3\mathrm{tr}_{12} - \mathrm{tr}_2^2\mathrm{tr}_{13} - \mathrm{tr}_1\mathrm{tr}_2\mathrm{tr}_{23} - 2\mathrm{tr}_1\mathrm{tr}_3 + 2\mathrm{tr}_{12}\mathrm{tr}_{23} + 4\mathrm{tr}_{13}, \\
& \mathrm{tr}_1^2\mathrm{tr}_2\mathrm{tr}_3 - \mathrm{tr}_1\mathrm{tr}_3\mathrm{tr}_{12} - \mathrm{tr}_1\mathrm{tr}_2\mathrm{tr}_{13} - \mathrm{tr}_1^2\mathrm{tr}_{23} - 2\mathrm{tr}_2\mathrm{tr}_3 + 2\mathrm{tr}_{12}\mathrm{tr}_{13} + 4\mathrm{tr}_{23}, \\
(3.9.1) \quad & \mathrm{tr}_1\mathrm{tr}_2\mathrm{tr}_3^2 - \mathrm{tr}_2\mathrm{tr}_3\mathrm{tr}_{13} - \mathrm{tr}_3^2\mathrm{tr}_{12} - \mathrm{tr}_1\mathrm{tr}_3\mathrm{tr}_{23} - 2\mathrm{tr}_1\mathrm{tr}_2 + 2\mathrm{tr}_{13}\mathrm{tr}_{23} + 4\mathrm{tr}_{12}).
\end{aligned}$$

We can interpret at least the equations 2-4 of this ideal by noting that for matrices in SL_2, we have

$$\mathrm{Tr}\left((M_1M_2)^2 - M_1^2M_2^2\right) = -\mathrm{tr}_1\mathrm{tr}_2\mathrm{tr}_{12} + \mathrm{tr}_1^2 + \mathrm{tr}_2^2 + \mathrm{tr}_{12}^2 - 4.$$

Other than that, what seems to happen is that in the coordinate ring of the singular locus, we get $\mathbb{C}[\mathrm{tr}_1, \mathrm{tr}_2, \mathrm{tr}_3]$ and then we get degree 2 extensions for the tr_{ij} from relations (2-4), with relations (5-10) adding some conditions that limit the degree of the overall extension.

For example, if one looks at the coordinate ring as an extension of

$$\mathbb{C}[\mathrm{tr}_1, \mathrm{tr}_2, \mathrm{tr}_3, \mathrm{tr}_{12}],$$

then at first glance equations (8-9) look like they can be solved uniquely for tr_{13} and tr_{23}; however, careful inspection gives that in fact the discriminant of the system of equations is a polynomial in (I), so this does not happen. We did not look into this matter further.

We have the following criterion for points to lie in the singular locus:

LEMMA 44. *A matrix triple $(A, B, C) \in SL_2^{\times 3}$ is in this singular locus (I) defined by (3.9.1) if and only if it is reducible.*

---

[1] See `http://www.math.uiuc.edu/Macaulay2/`; we used the `singularLocus` command to calculate the locus, and `decompose` to write it in a more recognizable form.

PROOF. We note that reducible matrix triples are in the singular locus. Conversely, take a triple of SL_2 matrices

$$\left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, C = \begin{pmatrix} i & j \\ k & l \end{pmatrix}\right)$$

in the singular locus.

First suppose one matrix, say A, can be diagonalized to $A = \mathrm{diag}(\lambda, \lambda^{-1})$. Looking at the second and fourth relations of (I), one finds that either $\lambda = \pm 1$, or $eh = il = 1$.

If $\lambda = \pm 1$, then the third relation is the only remaining one. Suppose we can diagonalize B, then this relation gives either $B = \pm \mathrm{Id}$ (so directly we have a reducible triple), or $il = 1$. But then because $il - jk = 1$, either $k = 0$ or $j = 0$ and we can swap the eigenvalues of B to see that the triple is reducible.

Otherwise, we have $eh = il = 1$, and so $fg = jk = 0$. If $gk = 0$ then we are done and if $fj = 0$ then we can swap the eigenvalues of A to see that the triple is reducible. If $gj = 0$, then the first relation gives $fk = 0$. This means that one of the B, C is diagonalizable: a case we already handled. The case $fk = 0$ is similar. This finishes the case when we have a diagonalizable matrix.

So now suppose neither matrix is diagonalizable. Writing A in Jordan form we get $a = b = d = 1, c = 0$. Relations 2 and 4 now give $g = k = 0$, so we are done. □

REMARK 45. • By Corollary 36, a matrix triple $(A, B, C) \in SL_2^{\times 3}$ is not stable if and only if it is reducible. Thus, the matrices in the singular locus correspond exactly to points in the embedding that are not stable.

• In [**4**, p.18-20], it is shown for general 2×2 matrices that (A, B) is reducible if and only if $\mathrm{Tr}(AB)^2 = \mathrm{Tr}A^2B^2$, and that (A, B, C) is reducible if and only if (A, B), (B, C) and (A, C) are reducible and $\mathrm{Tr}ABC = \mathrm{Tr}ACB$. This corresponds exactly to our above result.

We have now found a criterium for a matrix triple to be in the singular locus for the $SL_2^{\times 3}$ case: this corresponds to the point not being stable. Now, recall that a point being stable by definition means that its orbit is closed and the dimension of the orbit is maximal. The following two properties also holds between the various properties:

LEMMA 46. [**4**, p.25] *Let* $x = (M_1, ..., M_k) \in M_{2\times 2}^k$. *Then the orbit of* x *is closed if and only if either* $(M_1, ..., M_k)$ *is irreducible or* $(M_1, ..., M_k)$ *is simultaneously diagonalizable.*

PROOF. See [**4**, p.25]. □

LEMMA 47. *Let* $x = (M_1, M_2, ..., M_n) \in M_{2\times 2}^k$. *Then the orbit of* x *under simultaneous conjugation is smaller than 3 if and only if all matrices commute.*

PROOF. Suppose we know the that dimension of the orbit of one matrix, say M_1, under conjugation equals d. Then the orbit of (M_1, M_2) under conjugation is also d if and only if for any similarity matrix Q, the implication

$$QM_1Q^{-1} = M_1 \Rightarrow QM_2Q^{-1} = M_2 \tag{3.9.2}$$

holds. This is because should the dimension be larger than d, we need to have $\hat{Q}M_1\hat{Q}^{-1} = \tilde{Q}M_1\tilde{Q}^{-1}$ but $\hat{Q}M_2\hat{Q}^{-1} \neq \tilde{Q}M_2\tilde{Q}^{-1}$ for some $\hat{Q}$, $\tilde{Q}$. But conjugating with $\tilde{Q}^{-1}$ then gives $\tilde{Q}^{-1}\hat{Q}M_1\hat{Q}^{-1}\tilde{Q} = M_1$ but $\tilde{Q}^{-1}\hat{Q}M_2\hat{Q}^{-1}\tilde{Q} \neq M_2$, so (3.9.2) is broken for $Q = \tilde{Q}^{-1}\hat{Q}$.

Now, all matrices commuting means that either all matrices are simultaneously diagonalizable (if one matrix is diagonalizable, then for the other matrices to commute with it, they must be diagonalizable as well), or they can all be simultaneously conjugated to upper triangular form with ±1 on the diagonals (by the same argument).

In the first case, diagonalize the matrices; then they have common stabilizer $S_1 = \{\operatorname{diag}(\lambda, \lambda^{-1}) \mid \lambda \neq 0\}$; in the second case, the matrices have common stabilizer

$$S_2 = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathbb{C} \right\}; \tag{3.9.3}$$

in both cases the condition (3.9.2) clearly holds. Since the orbit of any one matrix is at most 2-dimensional (it is limited by the invariance of the trace and determinant), the orbit of the tuple must be at most 2-dimensional as well.

Conversely, suppose the orbit of some tuple x has dimension smaller than 3. Suppose one matrix, say M_1, can be diagonalized to $\operatorname{diag}(\lambda, \lambda^{-1})$ for some $\lambda \neq \pm 1$. Then already the orbit of x is 2. For the dimension of the orbit of x to be ≤ 2, then all other matrices must be stabilized by S_1. But as we know these matrices act as

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \lambda^2 b \\ \lambda^{-2} c & d \end{pmatrix},$$

so we must have $b = d = 0$, and all other matrices are diagonal matrices and commute with M_1.

Now suppose one matrix M_1 cannot be diagonalized, so its Jordan form is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The orbit of this matrix is also 2-dimensional: it consist of all matrices with trace 2 and determinant 1 other than the identity. Then for the dimension of the orbit of x to be ≤ 2, all other matrices must be stabilized by S_2. One checks that this means that all matrices must be upper triangular matrices with ±1 on the diagonal, commuting with M_1.

All this leaves is the trivial case of all matrices being equal to ±Id. This concludes the proof. □

Combining everything, we get:

PROPOSITION 48. *For any k-tuple of matrices in SL_2, we have the following set of equivalences and consequences:*

$$\begin{aligned} \neg\textit{stable} \Leftrightarrow \textit{reducible} \quad &\Leftrightarrow \quad \textit{commute} \vee \neg\textit{closed} \\ \textit{closed} \quad &\Leftrightarrow \quad \neg\textit{reducible} \vee \textit{simultaneously diagonalizable} \\ \textit{commute} \quad &\Leftrightarrow \quad \textit{dimension orbit} < 3 \\ \textit{simultaneously diag} \quad &\Rightarrow \quad \textit{commute.} \end{aligned}$$

For a 3-tuple, we know that reducibility is equivalent to the tuple being mapped to the singular locus of the algebra of invariants.

CHAPTER 4

The matrix space as a representation

We now look at our space of matrices with the SL_2 action in the general context of SL_2 representations. First, we very briefly describe how the representation theory of SL_2 works. Then we describe what happens in our situation.

4.1. Representation theory of SL_2

4.1.1. Complete reducibility. One main goal of representation theory is to classify, up to isomorphism, all possible representations of a given group. We introduce some basic notions.

If W is a subspace of V that is closed under G, then W is also representation for G: it is called a *subrepresentation* of V. A G-module V has at least two subrepresentations: the *trivial representation* $\{0\}$ and V itself. If these are all subrepresentations of V, then V is said to be *irreducible*.

A G-representation V is reducible if we can write it as a direct sum $V = W_1 \oplus W_2$ of G-representations (of course, $W_1 \cong W_1 \oplus \{0\}$ and $W_2 \cong \{0\} + W_2$ are then subrepresentations of G, so V is not irreducible). A G-representation V is called *completely reducible* if it can be written as a direct sum of irreducible G-representations.

If G satisfies certain conditions, then in this way we can write any finite-dimensional representation of G as a direct sum of irreducible representations. This way of writing is then also unique up to isomorphism. Representation theory, then, is concerned with determining, up to isomorphism, all these irreducible representations of a given group.

4.1.2. Irreducible representations of SL_2. The representation theory of SL_2 is well-known and quite simple. Any finite-dimensional SL_2-representation can be written as a unique direct sum of irreducible representations. Also, for any $0 \leq n \in \mathbb{N}$ there is a unique irreducible representation of dimension $n+1$, which is denoted $[n]$. This representation can be seen as

$$[n] = \{f \in \mathbb{C}[x_1, x_2] \,|\, f \text{ is homogeneous of degree } n\};$$

SL_2 acts on this in the following way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(x_1, x_2) = f(ax_1 + cx_2, bx_1 + dx_2).$$

When writing down direct sums of these irreducible representations, often the direct sum symbol $\oplus$ is denoted by $+$, and a direct sum of k times the irreducuble representation $[l]$ is often denoted as $k \cdot [l]$. For instance, we could write $[0] \oplus [0] \oplus [4] \oplus [6]$ as $2 \cdot [0] + [4] + [6]$.

Note that Hilbert's criterion for $[n]$ is very easy: we have

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \cdot x_1^k x_2^l = \lambda^{k-l} x_1^k x_2^l,$$

so:

PROPOSITION 49. *In the terminology of Mumford's criterion, the basis vector* $x_1^k x_2^{n-k}$ *of* $[n]$ *corresponds to a weight of* $2k-n$.

4.1.3. Some isomorphic representations. We now look at the action of SL_2 on vector spaces of matrices by simultaneous conjugation. We denote this action by $\diamond$, eg:

$$G \diamond \begin{pmatrix} a & b \\ c & d \end{pmatrix} = G \begin{pmatrix} a & b \\ c & d \end{pmatrix} G^{-1}.$$

PROPOSITION 50. *As a* SL_2 *representation by simultaneous conjugation* M_0 *is isomorphic to* $[2]$.

PROOF. One checks that the following linear map $\phi : M_0 \to [2]$ gives an isomorphism:

$$\begin{array}{rcl} M_2^0 & \cong & [2] \\ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} & \leftrightarrow & x_1^2 \\ \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \leftrightarrow & x_1 x_2 \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \leftrightarrow & x_2^2. \end{array}$$

For instance, one checks that

$$\begin{array}{rcl} \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \diamond \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\right) & = & \phi\begin{pmatrix} ac & -a^2 \\ c^2 & ac \end{pmatrix} = a^2 \cdot x_1^2 + 2ac \cdot x_1 x_2 + c^2 \cdot \phi x_2^2 \\ & = & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x_1^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \phi \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \end{array}$$

□

PROPOSITION 51. *The following isomorphism of* SL_2*-representations holds:*

$$[2] \otimes [2] \cong [0] + [2] + [4].$$

PROOF. We introduce the following notation for the basis elements:

$$[2] \otimes [2] = \operatorname{span}\{x_i \otimes x_j\}; \ [2] = \operatorname{span}\{y_1^2, y_1 y_2, y_2^2\}; \ [4] = \operatorname{span}\{z_1^k z_2^{4-k}\}.$$

We can then check that the following linear map $\phi : [0] + [2] + [4] \to [2] \otimes [2]$ defines an isomorphism of SL_2-representations:

$$\begin{array}{rcl} [0] + [2] + [4] & \cong & [2] \otimes [2] \\ 1 & \leftrightarrow & x_1^2 \otimes x_2^2 + x_2^2 \otimes x_1^2 - 2x_1 x_2 \otimes x_1 x_2 \\ y_1^2 & \leftrightarrow & x_1^2 \otimes x_1 x_2 - x_1 x_2 \otimes x_1^2 \\ y_1 y_2 & \leftrightarrow & \frac{1}{2}(x_1^2 \otimes x_2^2 - x_2^2 \otimes x_1^2) \\ y_2^2 & \leftrightarrow & x_1 x_2 \otimes x_2^2 - x_2^2 \otimes x_1 x_2 \\ z_1^4 & \leftrightarrow & x_1^2 \otimes x_1^2 \\ z_1^3 z_2 & \leftrightarrow & \frac{1}{2}(x_1^2 \otimes x_1 x_2 + x_1 x_2 \otimes x_1^2) \\ z_1^2 z_2^2 & \leftrightarrow & \frac{1}{6}(x_1^2 \otimes x_2^2 + 4 \cdot x_1 x_2 \otimes x_1 x_2 + x_2^2 \otimes x_1^2) \\ z_1 z_2^3 & \leftrightarrow & \frac{1}{2}(x_1 x_2 \otimes x_2^2 + x_2^2 \otimes x_1 x_2) \\ z_2^4 & \leftrightarrow & x_2^2 \otimes x_2^2. \end{array}$$

For instance, for a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ one checks that

$$\begin{aligned}
\phi\left(M \cdot y_1^2\right) &= \phi(a^2y_1^2 + 2acy_1y_2 + c^2y_2^2) \\
&= a^2\left(x_1^2 \otimes x_1x_2 - x_1x_2 \otimes x_1^2\right) + 2ac \cdot \frac{1}{2}(x_1^2 \otimes x_2^2 - x_2^2 \otimes x_1^2) + \\
&\quad c^2\left(x_1x_2 \otimes x_2^2 - x_2^2 \otimes x_1x_2\right) \\
&= (a^2bc - a^3d) \cdot x_1x_2 \otimes x_1^2 + (abc^2 - a^2cd)x_2^2 \otimes x_1^2 + \\
&\quad (a^3d - a^2bc)x_1^2 \otimes x_1x_2 + (bc^3 - ac^2d)x_2^2 \otimes x_1x_2 + \\
&\quad (a^2cd - abc^2)x_1^2 \otimes x_2^2 + (ac^2d - bc^3)x_1x_2 \otimes x_2^2 \\
&= M \cdot (x_1^2 \otimes x_1x_2 - x_1x_2 \otimes x_1^2) \\
&= M \cdot \phi(y_1^2).
\end{aligned}$$

The calculations with the $z_1^kz_2^{4-k}$ basis elements are longer but similar. □

4.2. Representation theory of Q

We are studying

$$Q = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) : \Delta \,|\, ad - bc = \Delta^2 \right\} \subset \mathbb{P}^4.$$

By definition, to find its stable and semi-stable points, we need to look at the SL_2 action on the corresponding affine space in $V = \mathbb{A}^5$. We now do this by looking at this space as a SL_2 representaton. But we can write $V = M_2^0 \oplus \mathbb{C} \oplus \mathbb{C}$, and SL_2 acts trivially on the $\mathbb{C}$'s, so by Proposition 50 we have the isomorphism of SL_2 representations

$$V \cong [2] \oplus [0] \oplus [0].$$

Now again by Lemma 31, a point of V is unstable if and only if there is a one-parameter subgroup $\mathrm{diag}(t, t^{-1})$ with all positive weights.

By Proposition 49, the only basis element of $[2] \oplus [0] \oplus [0]$ with positive weight is x_1^2, so the set of unstable points is $\{\lambda x_1^2 \,|\, \lambda \neq 0\}$; under our representation isomorphism this corresponds to the matrices

$$\left\{ \left(\begin{array}{cc} 0 & \lambda \\ 0 & 0 \end{array} \right) : 0 \,|\, \lambda \neq 0 \right\}.$$

This corresponds to our earlier results.

4.3. The $Q \times Q$-case

Similarly, $Q \times Q$ is a subset of $\mathbb{P}(V)$, with corresponding affine space V.

$$V = (M_2^0 \oplus \mathbb{C} \oplus \mathbb{C}) \otimes (M_2^0 \oplus \mathbb{C} \oplus \mathbb{C}).$$

4.3.1. An isomorphism of representations.

PROPOSITION 52. We have the following isomorphism of SL_2-representations:

$$V \cong [4] + 5 \cdot [2] + 5 \cdot [0].$$

PROOF. We have:

$$\begin{aligned}
V &\cong ([2] + 2 \cdot [0]) \otimes ([2] + 2 \cdot [0]) \\
&\cong [2] \otimes [2] + 2 \cdot [0] \otimes [2] + 2 \cdot [2] \otimes [0] + 4 \cdot [0] \otimes [0] \\
&\cong ([4] + [2] + [0]) + 2 \cdot [2] + 2 \cdot [2] + 4 \cdot [0] \\
&\cong [4] + 5 \cdot [2] + 5 \cdot [0];
\end{aligned}$$

here, we used Proposition 51 and the trivial representation isomorphisms $[0] \otimes V \cong V \cong V \otimes [0]$. □

Let us write out this isomorphism ϕ. Let us write the basis elements of $[4] + 5 \cdot [2] + 5 \cdot [0]$ as $x_1^k x_2^{4-k}$, $y_i^k z_i^{2-k}$ $(1 \leq i \leq 4)$, v_i $(1 \leq i \leq 5)$. Using our earlier calculations, we can find the elements of V to which these basis elements correspond. Let us denote the basis elements of $M_2^0 \oplus \mathbb{C} \oplus \mathbb{C}$ as

$$\left\{ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \Delta \right\}.$$

Then the basis $x_1^k x_2^{4-k}$ corresponds to the following matrices:

$$\begin{array}{rcl}
x_1^4 & \leftrightarrow & \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\
x_1^3 x_2 & \leftrightarrow & \dfrac{1}{2} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \dfrac{1}{2} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\
x_1^2 x_2^2 & \leftrightarrow & \dfrac{1}{6} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \dfrac{4}{6} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \dfrac{1}{6} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\
x_1 x_2^3 & \leftrightarrow & \dfrac{1}{2} \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \right) \\
x_2^4 & \leftrightarrow & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\end{array}$$

There is one [2]-module coming from $[2] \otimes [2]$:

$$\begin{array}{rcl}
y_1^2 & \leftrightarrow & \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\
y_1 z_1 & \leftrightarrow & \dfrac{1}{2} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \dfrac{1}{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}) \\
z_1^2 & \leftrightarrow & \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},
\end{array}$$

and four coming from $[2] \otimes [0]$ and $[0] \otimes [2]$. The basis elements $y_2^k z_2^{2-k}$ and $y_3^k z_3^{2-k}$ correspond to

$$\begin{array}{rcl}
y_{\{2,3\}}^2 & \leftrightarrow & \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \otimes \left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \Delta \right\} \\
y_{\{2,3\}} z_{\{2,3\}} & \leftrightarrow & \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \Delta \right\} \\
z_{\{2,3\}}^2 & \leftrightarrow & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \Delta \right\};
\end{array}$$

the other two are symmetric. Finally, we have 5 [0]-modules. Four of them are from the traces and determinants of the matrices:

$$v_{\{1,2,3,4\}} = \left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \Delta \right\} \otimes \left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \Delta \right\},$$

and the last one comes from the $[2] \otimes [2]$:

$$v_5 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - 2 \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

For example, one can check that this element of V is indeed invariant: let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then:

$$\begin{aligned}
M \diamond v_5 &= M \diamond \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) + M \diamond \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) - 2M \diamond \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} ac & -a^2 \\ c^2 & -ac \end{pmatrix} \otimes \begin{pmatrix} bd & -b^2 \\ d^2 & -bd \end{pmatrix} + \begin{pmatrix} bd & -b^2 \\ d^2 & -bd \end{pmatrix} \otimes \begin{pmatrix} ac & -a^2 \\ c^2 & -ac \end{pmatrix} \\
&\quad -2 \begin{pmatrix} \frac{1}{2}(ad+bc) & -ab \\ cd & -\frac{1}{2}(ad+bc) \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2}(ad+bc) & -ab \\ cd & -\frac{1}{2}(ad+bc) \end{pmatrix} \\
&= (a^2b^2 + a^2b^2 - 2a^2b^2) \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) + \\
&\quad (a^2bd + b^2ac - ab(ad+bc)) \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \\
&\quad (a^2d^2 + b^2c^2 - 2abcd) \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) + \ldots \\
&= 0 \cdot \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) + 0 \cdot \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + 1 \cdot \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) + \ldots \\
&= v_5.
\end{aligned}$$

4.3.2. Unstable points. Now we can find a basis for the set of unstable points of V: this is simply the union of the bases of the unstable points in the irreducible subrepresentations, for which we use Proposition 49. For the copy of [4] in V, this gives $\text{span}\{x_1^4, x_1^3x_2\}$, and for each copy of [2], we get $\text{span}\{y_i^2\}$. In all, we get a 7-dimensional subspace of V of unstable points. We note that

$$\begin{aligned}
\phi(x_1^3x_2) &= \frac{1}{2}\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\right), \\
\phi(y_1^2) &= \frac{1}{2}\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\right).
\end{aligned}$$

But then we can take the sum and difference of these basis elements to obtain:

PROPOSITION 53. *Up to conjugation, the unstable points of V are:*

$$\begin{aligned}
V^u &= \mathit{span}\left\{\phi(x_1^4), \phi(x_1^3x_2) + \phi(y_1^2), \phi(x_1^3x_2) - \phi(y_1^2), \phi(y_2^2), \phi(y_3^2), \phi(y_4^2), \phi(y_5^2)\right\} \\
&= \mathit{span}\{\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes \Delta, \\
&\qquad \Delta \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right), \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right), \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right)\}.
\end{aligned}$$

But note that this corresponds exactly to our results in Lemma 38.

4.3.3. Generalization to more matrices. When repeating the above calculations for more copies of Q, we need to repeat the argument of Proposition 52 to write the vector space in terms of irreducible representations. For example, for three Q's the calculation would look like:

$$\begin{aligned}
V &\cong ([2] + 2 \cdot [0]) \otimes ([2] + 2 \cdot [0]) \otimes ([2] + 2 \cdot [0]) \\
&\cong ([4] + 5 \cdot [2] + 5 \cdot [0]) \otimes ([2] + 2 \cdot [0]) \\
&\cong [4] \otimes [2] + 5 \cdot [2] \otimes [2] + 2 \cdot [4] + 10 \cdot [2] + 10 \cdot [0].
\end{aligned}$$

One see that this involves calculating more representation isomorphism as in Proposition 51. We did not look into this further, but suggest that using a computer algebra package might help with this task.

Conclusion, discussion, references

Conclusion

The object of this thesis was to study the object

$$\{(A_1, A_2, ..., A_m) \in SL_2^m\}/\sim,$$

where the equivalence was by simultaneous conjugation.

First, we looked at this problem with classical invariant theory. By constructing a suitable isomorphism, we reduced the calculation of $\mathbb{C}[SL_2^m]^{SL_2}$ to calculating $\mathbb{C}[M_0^m]^{SL_2}$ (Proposition 6). For the case $m = 2$, one can calculate $\mathbb{C}[M_0^m]^{SL_2}$ by elementary means (Proposition 8), and hence obtain $\mathbb{C}[SL_2^2]^{SL_2}$ (Corollary 11).

For general m, we studied the method to calculate $\mathbb{C}[M_0^m]^{SL_2}$ from [**2**], and worked out the proof in greater detail (Theorem 15). Again, we translated the results to $\mathbb{C}[SL_2^m]^{SL_2}$ (Corollary 17).

Next, to look at the problem geometrically, we embedded SL_2^m into a projective space $Q\times...\times Q$. We described a general method to cover the subset $(Q\times...\times Q)^{nn}$ of non-nilpotent matrices by open, affine, dense, SL_2-stable subsets U_a (section 2.4). We showed for the cases $m = 1$ (Proposition 22), $m = 2$ (Propositon 24) and $m = 3$ (Proposition 25) that this gave rise to a good quotient. We expect that this method will generalize to any m without problems.

Using Mumford's criterion, it is possible to formulate precise conditions under which points in $Q \times ... \times Q$ for any number of matrices m belong to the semi-stable and stable parts. This is done in Propositions 35 and 36. For $m > 1$ one then has that

$$(Q \times ... \times Q)^s \subsetneq (Q \times ... \times Q)^{nn} \subsetneq (Q \times ... \times Q)^{ss}.$$

In particular, the good quotient on the non-nilpotent matrices gives rise to a geometric quotient on the stable matrices.

In the cases $m = 2$ and $m = 3$ we showed that the map defining the good quotient on $(Q\times...\times Q)^{nn}$ in fact extends to a map on the semi-stable points. This could indicate that the map is in fact already a good quotient on the semi-stable points, and that this may in fact be the case for any m.

Using a computer algebra system, we calculated the singular locus of $\mathbb{C}[SL_2^3]^{SL_2}$. The defining ideal (3.9.1) at first glance looks pretty difficult to interpret. We were able to calculate exactly under what conditions matrices belong to this singular locus (Lemma 44). Some implications connecting the various concepts related to (semi-)stability of elements of SL_2^k are proved and summarized in Proposition 48.

One notes that the geometric invariant theory of SL_2 on the standard irreducible representations $[k]$ of SL_2 is very easy: see Proposition 49. But then by considering the affine space underlying $Q\times...\times Q$, one can calculate its stable/semi-stable points with representation theory. We checked for the semi-stable points for $m = 2$ that this gives the expected results (Proposition 53).

Discussion

As noted, it is an open question whether the good quotient on the non-nilpotent points extends to a quotient on all semi-stable points. This does seem to be the case. The reason is that we can obtain the good quotient for $(Q \times ... \times Q)^{ss}$ by calculating the invariants of the affine space $\mathbb{A}^{5^m}$ with relations $\Delta_i = a_i d_i - b_i c_i$ ([**9**, Theorem 6.4.4]). But we can just see the coordinate ring of this affine variety as a subset of the coordinate ring of $\mathbb{A}^{5m}$, so for example for $m = 2$ the invariant functions are those functions spanned by TrM_1, Δ_1, TrM_2, Δ_2,Tr$M_1 M_2$ that have the same degree in (M_1, Δ_1) as in (M_2, Δ_2): exactly the result that we found. Unfortunately we did not have enough time to process these insights into our main text.

In general, rather than looking at invariant functions on matrices, equivalently one can look at invariant functions on the irreducible representations. Associating $\mathbb{C}[V]$ with the symmetric algebra $S(V)$, one can then write $S^k(V)$ as a sum of irreducible representations, and the invariant functions of degree k are simply the trivial subrepresentations of $S^k(V)$. For instance, since $S^1(V) = V$, we know there are 5 independent linear invariant functions on $Q \times Q$ by the isomorphism $V \cong [4] + 5 \cdot [2] + 5 \cdot [0]$, and following the isomorphism one can in fact see that these are $\Delta_1 \Delta_2$, Tr$M_1 \Delta_2$, etcetera. Possibly applying techniques like these could give some more insights on our situation. It might also allow one to use standard computer algebra packages to perform calculations with.

Even though Proposition 8 is purely algebraic of nature, its proof uses some complex analysis. This is rather unelegant; it would be better to have a fully algebraic proof for this Proposition.

In Section 3.9, some work was done on the singular locus of the space generated by the invariants of SL_2^3. This gave rise to an ideal describing the singular locus. Possibly looking a bit more we could get some interesting results about the structure of the coordinate ring of the singular locus this ideal gives.

Finally, as noted, our choice to embed SL_2^m into projective space by adding m determinants was a heuristic one. In hindsight, in many ways one expects that the results obtained by just adding one determinant to embed SL_2^m in $\mathbb{P}^{4m}$ would be easier to calculate. In particular, it seems the unstable points are simply the nilpotent, commuting matrices and the non-stable points are the reducible ones. A good quotient should follow by calculating in the classic way the algebra of invariants of the corresponding affine algebraic variety in $\mathbb{A}^{4m+1} \supset \mathbb{P}^{4m}$.

References

There are two main works that are closely related to this thesis. Firstly, the article [**2**] calculates $\mathbb{C}[M_0^m]^{SL_2}$ and this calculation, which was repeated in this thesis, is central to the study of $\mathbb{C}[SL_2^m]^{SL_2}$. Secondly, the PhD thesis [**4**] is similar to our work in that it also tries to construct a geometric quotient of 2×2 matrices. In this latter work this is done for the affine space of tuples of 2×2 matrices, and the stable points are calculated directly rather than by using Mumford's criterion. Even so, the results obtained are quite similar and in some instances we used some of the calculations from that thesis.

In research into the calculation of invariants of larger matrices, usually representation theory is used, albeit in a rather different way than we did here. An elementary example in which representation theory is used to determine the generators of 3×3 matrices is [**1**]. It would be interesting to see how this calculation looks like when executed on 2×2 matrices. Finally, in the book [**3**], Chapter 5 gives a general introduction on the invariant theory of matrices, along with many references to articles on this topic.

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