Invariants under simultaneous conjugation of SL_2 matrices

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Outline

The problem

- Consider the space of m-tuples of matrices with determinant 1 (SL_2 matrices).
- Forget the basis: for any $Q \in SL_2$,

$$(M_1, \ldots, M_m) \sim (QM_1Q^{-1}, \ldots, QM_mQ^{-1}).$$

- We want to know the *orbit space:*

$$\{(M_1, \ldots, M_m) \in SL_2^m\} / \sim .$$

- This space arises in "monodromy of order 2 ordinary differential equations with $m+1$ singularities"

Approach 1: (Classical) Invariant Theory

- Well-known fact: trace, determinant of matrix are *invariant* under conjugation, so they "tell orbits apart"
- Invariant theory: classify these functions
- Notation: $\mathbb{C}[SL_2^m]$ are polynomial functions on $SL_2 \times \ldots \times SL_2$
- Notation: $\mathbb{C}[SL_2^m]^{SL_2}$ are polynomial functions invariant under conjugation

Our work

- For $\mathbb{C}[M_0^2]$, one can determine the invariants by elementary means
- In [Drensky2000], the complete structure of $\mathbb{C}[M_0^k]$ is calculated
- Proof uses classical results (1947) from the invariant theory of SO_3
- We wrote it down more clearly, ...
- ... and calculated $\mathbb{C}[SL_2^m]^{SL_2}$ by considering an isomorphism

$$\mathbb{C}[SL_2^m] \cong (\mathbb{C}[M_0^m] \otimes \mathbb{C}[t_1, \ldots, t_m])/(I).$$

Results: 2 matrices

Theorem (Algebra of invariants for $SL_2 \times SL_2$)

$$\mathbb{C}[SL_2 \times SL_2]^{SL_2} = \mathbb{C}[\mathrm{Tr}X_1, \mathrm{Tr}X_2, \mathrm{Tr}X_1X_2].$$

- The algebra of invariants is generated by these three functions
- The three functions are algebraically independent

Results: 3 matrices

Theorem (Algebra of invariants for $SL_2 \times SL_2 \times SL_2$)

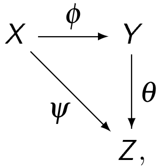
Let $a = \mathrm{Tr} x_1$, $b = \mathrm{Tr} x_2$, $c = \mathrm{Tr} x_3$, $d = \mathrm{Tr} x_1 x_2$, $e = \mathrm{Tr} x_1 x_3$, $f = \mathrm{Tr} x_2 x_3$, $g = \mathrm{Tr} x_1 x_2 x_3 - \mathrm{Tr} x_1 x_3 x_2$. Then $\mathbb{C}[SL_2^3]^{SL_2} = \mathbb{C}[a,b,c,d,e,f,g]/(rel)$, where

$$\begin{aligned} \text{rel} &= g^2 + 4a^2 + 4b^2 + 4c^2 + 4d^2 + 4e^2 + 4f^2 \\ &\quad + 2a^2bcf + 2abc^2d + 2ab^2ce \\ &\quad - 4ace - 4abd - 4bcf + 4def \\ &\quad - b^2e^2 - a^2f^2 - c^2d^2 \\ &\quad - 2bcde - 2acdf - 2abef - a^2b^2c^2 - 16. \end{aligned}$$

- Or: $\mathbb{C}[SL_2^3]^{SL_2} = \mathbb{C}[a,b,c,d,e,f] + \mathbb{C}[a,b,c,d,e,f] \cdot g$

Approach 2: Geometric Invariant Theory

- Projective geometry of $X = \{(M_1, ..., M_m) \in SL_2^m\}/\sim$
- Want: Y that separates the orbits as much as possible: every map ψ that is constant on orbits "runs through Y":



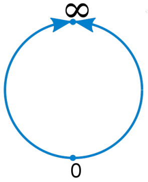
(the "universal mapping property")

- Called *good quotient*
- Called *geometric quotient* if it splits the orbits completely

Projective space: an introduction

Consider the space $\mathbb{R} \cup \{\infty\}$. Denote a point by $(a:b)$, where $(a:b) = (\lambda a : \lambda b)$

- $x \in \mathbb{R}$ corresponds to $(x:1)$
- ∞ corresponds to $(1:0)$
- $(0:0)$ is excluded
- Geometry on projective spaces is much more elegant: in the projective plane every two lines intersect and other nice properties.
- Generalize to:



$$\mathbb{P}^n = \{(x_1 : \ldots : x_{n+1}) \in \mathbb{C}^{n+1}\} \backslash (0 : \ldots : 0)_{/(x_1:\ldots:x_{n+1})=(\lambda x_1:\ldots:\lambda x_{n+1})}.$$

Embedding SL_2 in a projective space

Obvious choice: add determinant as extra coordinate: let

$$Q = \{(a:b:c:d:\Delta) \subset \mathbb{P}^4 \,|\, ad - bc = \Delta^2\},$$

and define embedding

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{\phi}{\hookrightarrow} (a:b:c:d:1) = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : 1\right).$$

The conjugation can be extended to the whole of Q: conjugating an element by $M \in SL_2$ gives:

$$\left(M \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) M^{-1} : \Delta\right).$$

Embedding SL_2^m in a projective space

We can send tuples of matrices to tuples of Q's, e.g.:

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \hookrightarrow \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} : \Delta_1, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} : \Delta_2\right).$$

This is a projective space by considering this as an embedding into $\mathbb{P}^{24}$:

$$\begin{array}{rcl} Q \times Q & \hookrightarrow & \mathbb{P}^{24} \\ \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} : \Delta_1, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} : \Delta_2\right) & \mapsto & (a_1 a_2 : a_1 b_2 : a_1 c_2 : a_1 d_2 : \\ & & a_1 \Delta_2 : \ldots : \Delta_1 a_2 : \Delta_1 b_2 : \\ & & \Delta_1 c_2 : \Delta_1 d_2 : \Delta_1 \Delta_2). \end{array}$$

Again, conjugation is as an action on the complete $\mathbb{P}^{24}$.

Constructing the quotient

- For affine spaces, the map *induced* by the algebra of invariants is a good quotient.
- For example: $X = SL_2 \times SL_2$, $Y = \mathbb{C}^3$:

$$\phi(M_1, M_2) = (\mathrm{Tr} M_1, \mathrm{Tr} M_2, \mathrm{Tr} M_1 M_2).$$

- Idea: write

$$Q \times Q = \bigcup_{\alpha \in A} U_\alpha :$$

  a *cover* by *open*, *affine*, *dense*, SL_2*-stable* subsets.
- If we "glue together" these quotients, we get a new quotient.

Glueing a quotient

- For four affine parts, we can make quotients mapping $((M_1 : \Delta_1),(M_2 : \Delta_2))$ to:

$$(\mathrm{Tr} M_1 \Delta_2 : \mathrm{Tr} M_2 \Delta_1 : \mathrm{Tr} M_1 M_2 : \Delta_1 \Delta_2);$$
$$(\mathrm{Tr} M_1 \mathrm{Tr} M_2 : \mathrm{Tr} M_1 M_2 : \Delta_1 \Delta_2 : \mathrm{Tr} M_1 \Delta_2);$$
$$(\mathrm{Tr} M_1 \mathrm{Tr} M_2 : \mathrm{Tr} M_1 M_2 : \Delta_1 \Delta_2 : \mathrm{Tr} M_2 \Delta_1);$$
$$(\mathrm{Tr} M_2 \Delta_1 : \mathrm{Tr} M_1 \Delta_2 : \mathrm{Tr} M_1 M_2 : \mathrm{Tr} M_1 \mathrm{Tr} M_2).$$

- Combination $Y = \{(a : b : c : d : e) \mid ab = cd\} \setminus \{(0 : \ldots : 0 : 1)\}$; ϕ maps $((M_1 : \Delta_1),(M_2 : \Delta_2))$ to

$$(\mathrm{Tr} M_1 \Delta_2 : \mathrm{Tr} M_2 \Delta_1 : \mathrm{Tr} M_1 \mathrm{Tr} M_2 : \Delta_1 \Delta_2 : \mathrm{Tr} M_1 M_2).$$

- (Y, ϕ) is a good quotient for the set $(Q \times Q)^{nn}$ of non-nilpotent matrices.

Stable and semi-stable points

- One can determine $(Q \times Q)^s$: *stable* points and $(Q \times Q)^{ss}$: *semi-stable* points. Then $(Q \times Q)^{ss} \supsetneq (Q \times Q)^{nn} \supsetneq (Q \times Q)^s$.

Theorem (the general theory)

There is a good quotient for the semi-stable points. This is a geometric quotient for the stable points.

Theorem (our situation)

ϕ is a geometric quotient from $(Q \times Q)^s$ to

$$\{(x_1 : x_2 : x_3 : x_4 : x_5) \,|\, x_1 x_2 = x_3 x_4, x_1^2 + x_2^2 + x_5^2 - x_3 x_5 - 4x_4^2 \neq 0\}.$$

- Probably: $\phi : (Q \times Q)^{ss} \rightarrow \{(a : b : c : d : e) \,|\, ab = cd\}$ is a good quotient

Approach 3: Representation Theory

- More abstractly, the action of SL_2 on the set of matrices can be seen as a *representation* of SL_2
- Idea: forget the specific space SL_2 acts on
- Representation theory: classify all possible actions
- Every representation of SL_2 is a sum of *irreducible representations*
- The irreducible representations of SL_2 are $[k]$ of dimension $k+1$

Results

- For the 25-coordinate space that $Q \times Q$ was embedded in, one can make a correspondence

$$V \cong [4] + 5 \cdot [2] + 5 \cdot [0].$$

- Verify that we get the same semi-stable points.
- For bigger calculations, use computer algebra packages?

Conclusion & Evaluation

- In the affine case, $\mathbb{C}[SL_2^m]^{SL_2}$ was determined
- This gives rise to a good quotient on the projective

$$(Q \times \ldots \times Q)^{ss} \supsetneq (Q \times \ldots \times Q)^{nn} \supsetneq (Q \times \ldots \times Q)^{s}$$

- Semi-simple and stable points can also be interpreted with representation theory.

- Good quotient on $(Q \times \ldots \times Q)^{ss}$? Probably: yes.
- Use another embedding?

Literature

W. Drensky.
Defining Relations for the Algebra of Invariants of 2×2 Matrices.
Algebras and Representation Theory, 2003.

A. Extra.
The invariants of 2×2 matrices, their algebraic relations and the corresponding moduli problem.
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See `http://meilof.home.fmf.nl/scriptie/` for (the draft of) my thesis.